SQUAREFREE P-MODULES AND THE cd-INDEX

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Abstract. In this paper, we introduce a new algebraic concept, which we call squarefree P-modules. This concept is inspired from Karu's proof of the non-negativity of the **cd**-indices of Gorenstein* posets, and supplies a way to study **cd**-indices from the viewpoint of commutative algebra. Indeed, by using the theory of squarefree P-modules, we give several new algebraic and combinatorial results on CW-posets. First, we define an analogue of the **cd**-index for any CW-poset and prove its non-negativity when a CW-poset is Cohen–Macaulay. This result proves that the h-vector of the barycentric subdivision of a Cohen–Macaulay regular CW-complex is unimodal. Second, we prove that the Stanley–Reisner ring of the barycentric subdivision of an odd dimensional Cohen–Macaulay polyhedral complex has the weak Lefschetz property. Third, we obtain sharp upper bounds of the **cd**-indices of Gorenstein* posets for a fixed rank generating function.

1. Introduction

After Stanley [St1] gave a beautiful proof of the upper bound theorem for triangulated spheres by using Reisner's criterion [Rei], the study of Stanley–Reisner rings and its applications to f-vector theory have been of great interest both in combinatorics and in commutative algebra. In this paper, we introduce a new algebraic concept to study flag f-vectors of finite posets, which we call squarefree P-modules, and consider its applications.

Squarefree P-modules are defined as an analogue of squarefree modules [Ya1] which are module-theoretic generalization of Stanley–Reisner rings. The concept of squarefree P-modules is inspired from the work of Karu [Ka] who proved the non-negativity of the **cd**-indices of Gorenstein* posets by using sheaves of finite vector spaces on posets. Indeed, we show that there is a one-to-one correspondence between the squarefree P-modules and the sheaves on a poset P, and that squarefree P-modules give a way to interpret Karu's proof of the non-negativity of **cd**-indices in terms of commutative algebra. We give the definition of squarefree P-modules in Section 2, and study their basic algebraic properties in the first half of the paper.

In the latter half of the paper, we consider applications of squarefree P-modules to f-vector theory, particularly to the study of **cd**-indices. First, we quickly review the theory of **cd**-indices. We refer the readers to [St4, Section 3] for basics on the theory of partially ordered sets. Let P be a finite partially ordered set (poset) of rank n with the minimal elements $\hat{0}$. The *order complex* Δ_P of P is the abstract simplicial complex whose faces are chains of $P \setminus \{\hat{0}\}$, namely,

$$\Delta_P = \{\{\sigma_1, \dots, \sigma_k\} \subset P \setminus \{\hat{0}\} : \sigma_1 < \cdots < \sigma_k\}.$$

(In this paper, we assume that all posets have the minimal element, but we ignore it when we consider their order complexes.) Note that the rank of P is the maximal cardinality of elements of Δ_P. For a finite set X, we write $|X|$ for its cardinality. For a subset $S \subset [n] = \{1, \dots, n\}$, an element $C = \{\sigma_1, \dots, \sigma_k\} \in \Delta_P$ is called an *S-chain* if $\{\operatorname{rank} \sigma_1, \dots, \operatorname{rank} \sigma_k\} = S$, where we define $\operatorname{rank} \sigma = \max\{|C| : C \in \Delta_P, \max C = \sigma\}$ for $\sigma \neq \hat{0}$ and $\operatorname{rank} \hat{0} = 0$. Let $f_S(P)$ be the number of S-chains of P. Define $h_S(P)$ by

$$h_S(P) = \sum_{T \subseteq S} (-1)^{|S|-|T|} f_T(P).$$

The vectors $(f_S(P) : S \subset [n])$ and $(h_S(P) : S \subset [n])$ are called the *flag f-vector* and the *flag h-vector* of P respectively.

We express the flag h-vector of P by the homogeneous non-commutative polynomial, called the **ab**-index of P. For a subset $S \subset [n]$, its *characteristic monomial* is the non-commutative monomial $w_S = w_1 w_2 \cdots w_n$ in variables $\mathbf{a}$ and $\mathbf{b}$ defined by $w_i = \mathbf{a}$ if $i \notin S$ and $w_i = \mathbf{b}$ if $i \in S$. The **ab**-*index* of P is the polynomial

$$\Psi_P(\mathbf{a}, \mathbf{b}) = \sum_{S \subset [n]} h_S(P) w_S \in \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle,$$

where $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ denotes the non-commutative polynomial ring over $\mathbb{Z}$ with the variables $\mathbf{a}$ and $\mathbf{b}$. Now we recall the definition of the **cd**-index. We say that P is Gorenstein* if Δ_P is a Gorenstein* simplicial complex (see [St3, p. 67]). If P is Gorenstein* then there is a non-commutative polynomial $\Phi_P(\mathbf{c}, \mathbf{d}) \in \mathbb{Z}\langle \mathbf{c}, \mathbf{d} \rangle$ in the variables $\mathbf{c}$ and $\mathbf{d}$ such that $\Phi_P(\mathbf{a} + \mathbf{b}, \mathbf{ab} + \mathbf{ba}) = \Psi_P(\mathbf{a}, \mathbf{b})$ (see [St4, Theorem 3.17.1]). This polynomial $\Phi_P(\mathbf{c}, \mathbf{d})$ is called the **cd**-*index* of P. Note that $\Phi_P(\mathbf{c}, \mathbf{d})$ is homogeneous of degree n in the grading $\deg \mathbf{c} = 1$ and $\deg \mathbf{d} = 2$. In this paper, to simplify the notation, we regard $\Phi_P(\mathbf{c}, \mathbf{d})$ as a polynomial in $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ by the identifications $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$.

The **cd**-index has two important properties. First, it efficiently encodes the flag f-vectors of Gorenstein* posets. The flag f-vectors of posets of rank n have 2^n entries. However, the **cd**-polynomial of degree n is a linear combination of the $(n+1)$th Fibonacci number F_{n+1} ($F_1 = F_2 = 1, F_{k+2} = F_{k+1} + F_k$) of monomials, which is much smaller than 2^n. Moreover, it is known that the existence of the **cd**-index describes all linear equations satisfied by the flag f-vectors of Gorenstein* posets [BB, BK]. Another important property of the **cd**-index of a Gorenstein* poset is its non-negativity, which was proved by Karu [Ka].

Our first result is an extension of the notion of the **cd**-index to CW-posets. Let P be a finite poset with the minimal element $\hat{0}$. We say that P is a *quasi CW-poset* if, for every $\sigma \in P \setminus \{\hat{0}\}$, the poset $\partial\sigma = \{\tau \in P : \tau < \sigma\}$ is Gorenstein*, and that P is a *CW-poset* if, for every $\sigma \in P \setminus \{\hat{0}\}$, the geometric realization of $\Delta_{\partial\sigma}$ is homeomorphic to a sphere. Note that CW-posets and Gorenstein* posets are quasi CW-posets. Also, since a finite poset is a CW-poset if and only if it is the face poset of a finite regular CW-complex [Bj], considering CW-posets is equivalent to considering finite regular CW-complexes. A finite poset P is said to be *Cohen–Macaulay* if the simplicial complex Δ_P is a Cohen–Macaulay simplicial complex (see [St3, p. 58]).

Theorem 1.1. *Let P be a quasi CW-poset of rank n. There are unique* **cd**-*polynomials $\Phi^{\mathbf{d}}, \Phi^{\mathbf{a}}, \Phi^{\mathbf{b}} \in \mathbb{Z}\langle \mathbf{c}, \mathbf{d}\rangle$ such that*

$$\Psi_P(\mathbf{a},\mathbf{b}) = \Phi^{\mathbf{d}} \cdot \mathbf{d} + \Phi^{\mathbf{a}} \cdot \mathbf{a} + \Phi^{\mathbf{b}} \cdot \mathbf{b}. \tag{1}$$

Moreover, if P is Cohen–Macaulay then all the coefficients in $\Phi^{\mathbf{d}}, \Phi^{\mathbf{a}}$ and $\Phi^{\mathbf{b}}$ are non-negative.

Since $\Phi^{\mathbf{d}}, \Phi^{\mathbf{a}}, \Phi^{\mathbf{b}}$ are homogeneous of degrees $n-2, n-1$ and $n-1$ respectively, Theorem 1.1 gives a way to express flag f-vectors of CW-posets by $(n+2)$th Fibonacci number of integers. We prove in Proposition 5.2 that the existence of the expression (1) indeed describes all linear equations satisfied by the flag f-vectors of (Cohen–Macaulay) CW-posets. Also, the non-negativity statement for Cohen–Macaulay quasi CW-posets implies an interesting result for ordinary h-vectors. For a poset P of rank n, the *h-vector* $h(\Delta_P) = (h_0, h_1, \dots, h_n)$ of Δ_P is given by $h_i = \sum_{S \subset [n], |S|=i} h_S(P)$. We say that a vector $(h_0, h_1, \dots, h_n) \in \mathbb{Z}^{n+1}$ is *unimodal* if $h_0 \leq h_1 \leq \cdots \leq h_p \geq \cdots \geq h_n$ for some integer $0 \leq p \leq n$. By using Theorem 1.1 we prove

Corollary 1.2. *If P is a Cohen–Macaulay quasi CW-poset then the h-vector of Δ_P is unimodal.*

Recall that if P is a CW-poset, then Δ_P is combinatorially isomorphic to the barycentric subdivision of a regular CW-complex corresponding to P. It was proved by Brenti and Welker [BW, Corollary 3] that the h-vector of the barycentric subdivision of a Cohen–Macaulay Boolean cell complex is unimodal. Corollary 1.2 says that this unimodality result holds at the level of regular CW-complexes.

In f-vector theory, once we have an unimodal sequence $h_0 \leq \cdots \leq h_p \geq \cdots \geq h_n$, it is natural to ask if the sequence $(h_0, h_1 - h_0, \dots, h_p - h_{p-1})$ has a nice property. We study this problem for the h-vectors of the barycentric subdivisions of polyhedral complexes. We say that a CW-poset P is *of polyhedral type* if, for every $\sigma \in P \setminus \{\hat{0}\}$, the subposet $\langle \sigma \rangle = \{\tau \in P : \tau \leq \sigma\}$ is the face poset of a convex polytope. Let $\lfloor x \rfloor$ denote the integer part of $x \in \mathbb{Q}$.

Theorem 1.3. *Let P be a Cohen–Macaulay CW-poset of polyhedral type having rank n and let $h(\Delta_P) = (h_0, h_1, \dots, h_n)$ be the h-vector of Δ_P. Then the vector $(h_0, h_1 - h_0, \dots, h_{\lfloor \frac{n}{2} \rfloor} - h_{\lfloor \frac{n}{2} \rfloor - 1})$ is the f-vector of a simplicial complex.*

To prove the above theorem, we study the weak Lefschetz property (WLP for short) of squarefree P-modules. See Section 6 for the definition of the WLP. Indeed, we prove that the order complex of a Cohen–Macaulay CW-poset of polyhedral type has the WLP over $\mathbb{R}$ if its rank is even (Corollary 6.3). This result gives a partial solution to the conjecture of Kubitzke and Nevo [KN, Conjecture 4.12] who conjectured that the barycentric subdivision of a Cohen–Macaulay simplicial complex has the WLP.

Our final result is about upper bounds of the **cd**-indices of Gorenstein* posets for a fixed rank generating function. To state the result, we introduce a way to describe **cd**-monomials by certain subsets. Let $\mathcal{A}_n$ be the set of subsets of $[n-1]$ which contains no consecutive integers, namely,

$$\mathcal{A}_n = \{S \subset [n-1] : \{i, i+1\} \not\subset S \text{ for } i = 1, 2, \dots, n-2\}.$$

Let $\mathcal{B}_n$ be the set of $\mathbf{cd}$-monomials of degree n. Then there is a bijection $\kappa_n : \mathcal{B}_n \to \mathcal{A}_n$ defined by

$$\kappa_n(\mathbf{c}^{s_0}\mathbf{d}\mathbf{c}^{s_1}\mathbf{d}\cdots\mathbf{d}\mathbf{c}^{s_k}) = \{s_0+1, s_0+s_1+3, \dots, s_0+\cdots+s_{k-1}+2k-1\}.$$

For a $\mathbf{cd}$-polynomial $\Phi = \sum_{v\in\mathcal{B}_n} \alpha_v v$ of degree n, where $\alpha_v \in \mathbb{Z}$, and for any $S \in \mathcal{A}_n$, let $\alpha_S(\Phi) = \alpha_{\kappa_n^{-1}(S)}$. Thus, if we write $\alpha_S = \alpha_S(\Phi)$ and $v^S = \kappa_n^{-1}(S)$, then we have $\Phi = \sum_{S\in\mathcal{A}_n} \alpha_S v^S$. For example, a $\mathbf{cd}$-polynomial of degree 4 will be written in the form $\Phi = \alpha_\emptyset \mathbf{c}^4 + \alpha_{\{1\}}\mathbf{d}\mathbf{c}^2 + \alpha_{\{2\}}\mathbf{c}\mathbf{d}\mathbf{c} + \alpha_{\{3\}}\mathbf{c}^2\mathbf{d} + \alpha_{\{1,3\}}\mathbf{d}^2$. For a Gorenstein* poset P, we define $\alpha_S(P) = \alpha_S(\Phi_P(\mathbf{c},\mathbf{d}))$. We prove the following bound for $\alpha_S(P)$.

Theorem 1.4. *If P is a Gorenstein* poset of rank n, then $\alpha_S(P) \le \prod_{i\in S} \alpha_{\{i\}}(P)$ for all $S \in \mathcal{A}_n$.*

The upper bounds in Theorem 1.4 are sharp. Indeed, we prove in Proposition 7.10 that for any sequence $\alpha_1, \dots, \alpha_{n-1}$ of non-negative integers there is a Gorenstein* poset P of rank n such that $\alpha_S(P) = \prod_{i\in S}\alpha_i$ for all $S \in \mathcal{A}_n$. Also, since knowing $\alpha_{\{1\}}(P), \dots, \alpha_{\{n-1\}}(P)$ is equivalent to knowing $f_{\{1\}}(P), \dots, f_{\{n\}}(P)$ (see Section 7), Theorem 1.4 gives sharp upper bounds of the $\mathbf{cd}$-indices of Gorenstein* posets for a fixed rank generating function, and therefore gives sharp upper bounds of the flag h-vectors of Gorenstein* posets for a fixed rank generating function.

This paper is organized as follows: In Section 2, we define squarefree P-modules and study their basic algebraic properties. In Section 3 and 4, we study homological properties of squarefree P-modules. In Section 5, we translate Karu's proof of the non-negativity of $\mathbf{cd}$-indices in terms of squarefree P-modules, and prove Theorem 1.1 and Corollary 1.2. In Section 6, we study the weak Lefschetz property of Cohen–Macaulay squarefree P-modules, and prove Theorem 1.3. In Section 7, upper bounds of the $\mathbf{cd}$-indices of Gorenstein* posets for a fixed rank generating function are studied. In Section 8, we present some open problems that arose during this research.

2. Squarefree P-modules

Throughout the paper, we assume that every poset is finite and has the minimal element $\hat{0}$. For a poset P and $\sigma \in P$, we write $\widehat{P} = P \setminus \{\hat{0}\}$, $\langle\sigma\rangle = \{\tau \in P : \tau \le \sigma\}$ and $\partial\sigma = \langle\sigma\rangle \setminus \{\sigma\}$. We denote by $\mathbb{N}$ the set of non-negative integers.

In the rest of this section, we fix a poset P of rank n.

Squarefree P-modules. Let $R = K[x_\sigma : \sigma \in \widehat{P}]$ be the polynomial ring over a field K whose variables are indexed by the elements of $\widehat{P}$. We consider the $\mathbb{N}^{|\widehat{P}|}$-grading of R by defining that the degree of x_σ is $\mathbf{e}_\sigma \in \mathbb{N}^{|\widehat{P}|}$, where $\{\mathbf{e}_\sigma : \sigma \in \widehat{P}\}$ is the standard basis of $\mathbb{N}^{|\widehat{P}|}$. For $\mathbf{u} = \sum_{\sigma\in\widehat{P}} u_\sigma \mathbf{e}_\sigma \in \mathbb{N}^{|\widehat{P}|}$, let $\operatorname{supp}(\mathbf{u}) = \{\sigma \in \widehat{P} : u_\sigma \ne 0\}$. For an $\mathbb{N}^{|\widehat{P}|}$-graded R-module M, let $M_\mathbf{u}$ be its graded component of degree $\mathbf{u} \in \mathbb{N}^{|\widehat{P}|}$. The polynomial ring R also has a standard $\mathbb{Z}$-graded structure defined by $\deg x_\sigma = 1$ for any $\sigma \in \widehat{P}$. Any $\mathbb{N}^{|\widehat{P}|}$-graded R-module has this $\mathbb{Z}$-graded structure. We will sometimes consider this standard $\mathbb{Z}$-grading.

Definition 2.1. A *squarefree P-module* (over K) is a finitely generated $\mathbb{N}^{|\widehat{P}|}$-graded R-module M satisfying the following two conditions:

(a) For any $\mathbf{u} \in \mathbb{N}^{|\widehat{P}|}$ with $\mathrm{supp}(\mathbf{u}) \notin \Delta_P$, one has $M_{\mathbf{u}} = 0$.

(b) For any $\mathbf{u} \in \mathbb{N}^{|\widehat{P}|}$ with $\mathrm{supp}(\mathbf{u}) \in \Delta_P$ and for any $\tau \in P$, the multiplication

$$\times x_\tau : M_{\mathbf{u}} \to M_{\mathbf{u}+\mathbf{e}_\tau}$$

is bijective if $\mathrm{supp}(\mathbf{u}+\mathbf{e}_\tau) \in \Delta_P$ and $\tau \leq \max(\mathrm{supp}(\mathbf{u}))$.

A typical example of a squarefree P-module is the Stanley–Reisner ring of Δ_P. Recall that, for a finite abstract simplicial complex Δ with the vertex set V, its *Stanley–Reisner ring* (over K) is the ring

$$K[\Delta] = K[x_v : v \in V]/(\textstyle\prod_{v\in F} x_v : F \subset V, F \notin \Delta).$$

For simplicity, we write $K[Q] = K[\Delta_Q]$ for a finite poset Q. The Stanley–Reisner ring $K[P]$ is a squarefree P-module since $(K[P])_{\mathbf{u}}$ is the 1-dimensional K-vector space spanned by the monomial $\prod_{\sigma\in\widehat{P}} x_\sigma^{u_\sigma}$ if $\mathrm{supp}(\mathbf{u}) \in \Delta_P$ and is zero if $\mathrm{supp}(\mathbf{u}) \notin \Delta_P$. Also, it is easy to see that, if Q is an order ideal of P, that is, if Q is a subposet of P satisfying that $\sigma \in Q$ and $\tau < \sigma$ imply $\tau \in Q$, then $K[Q]$ is a squarefree P-module (by regarding $K[Q]$ as an R-module). In particular, for any $\sigma \in \widehat{P}$, $K[\langle\sigma\rangle]$ and $K[\partial\sigma]$ are squarefree P-modules.

In this paper, we say that a poset Q is *Cohen–Macaulay* (resp. *Gorenstein**) over a field K if $K[Q]$ is Cohen–Macaulay (resp. Gorenstein*). Also, when we consider the Cohen–Macaulay property (or Gorenstein* property) we skip the condition on a field K if it is arbitrary. Note that if Q is Cohen–Macaulay over some field then it is Cohen–Macaulay over $\mathbb{R}$.

In the rest of this section, we discuss basic properties of squarefree P-modules.

Maps of squarefree P-modules. A *map of squarefree P-modules* is a degree preserving $\mathbb{N}^{|\widehat{P}|}$-graded R-homomorphism between squarefree P-modules.

Lemma 2.2. *If $\varphi : N \to M$ is a map of squarefree P-modules, then* $\ker\varphi, \mathrm{im}\,\varphi$ *and* $\mathrm{coker}\,\varphi$ *are squarefree P-modules.*

Proof. It is clear that $\ker\varphi, \mathrm{im}\,\varphi$ and $\mathrm{coker}\,\varphi$ satisfy condition (a) of squarefree P-modules. To see that they also satisfy condition (b), consider the following commutative diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow & 0 & \longrightarrow & (\ker\varphi)_{\mathbf{u}} & \longrightarrow & N_{\mathbf{u}} & \xrightarrow{\varphi} & M_{\mathbf{u}} \\
 & \downarrow & & \downarrow \times x_\tau & & \downarrow \times x_\tau & & \downarrow \times x_\tau \\
0 \longrightarrow & 0 & \longrightarrow & (\ker\varphi)_{\mathbf{u}+\mathbf{e}_\tau} & \longrightarrow & N_{\mathbf{u}+\mathbf{e}_\tau} & \xrightarrow{\varphi} & M_{\mathbf{u}+\mathbf{e}_\tau},
\end{array}$$

The above diagram and the five lemma imply that $\ker\varphi$ satisfies condition (b). The proofs for $\mathrm{im}\,\varphi$ and $\mathrm{coker}\,\varphi$ are similar. □

Krull dimensions. Condition (b) of squarefree P-modules is equivalent to the following condition:

(b') For any $\sigma \in \widehat{P}$ and for any monomial $x^{\mathbf{u}} = \prod_{\rho\in\widehat{P}} x_\rho^{u_\rho} \in R$ which is non-zero in $K[\langle\sigma\rangle]$, the multiplication $\times x^{\mathbf{u}} : M_{\mathbf{e}_\sigma} \to M_{\mathbf{e}_\sigma+\mathbf{u}}$ is bijective.

This implies a useful decomposition formula of squarefree P-modules. For convention, we write $\mathbf{e}_{\hat{0}} = \mathbf{0}$ for the zero vector in $\mathbb{N}^{|\widehat{P}|}$ and write $K[\langle\hat{0}\rangle] = K[\{\hat{0}\}] = K$.

Lemma 2.3. *If M is a squarefree P-module, then as K-vector spaces one has*

$$M \cong \bigoplus_{\sigma \in P} (M_{\mathbf{e}_\sigma} \otimes_K K[\langle \sigma \rangle]) .$$

Proof. Let $N = \bigoplus_{\sigma \in P} (M_{\mathbf{e}_\sigma} \otimes_K K[\langle \sigma \rangle])$. We claim that $N_{\mathbf{u}} \cong M_{\mathbf{u}}$ for all $\mathbf{u} \in \mathbb{N}^{|\widehat{P}|}$. The claim is obvious when $\mathbf{u}$ is the zero vector. Also, by condition (a) of squarefree P-modules, we may assume $\mathrm{supp}(\mathbf{u}) \in \Delta_P$. Let $\mathbf{u} \in \mathbb{N}^{|\widehat{P}|}$ with $\mathrm{supp}(\mathbf{u}) = \{\sigma_1, \dots, \sigma_k\} \in \Delta_P$, where $\sigma_k = \max(\mathrm{supp}(\mathbf{u}))$. Then we have

$$N_{\mathbf{u}} = (M_{\mathbf{e}_{\sigma_k}} \otimes_K K[\langle \sigma_k \rangle])_{\mathbf{u}} \cong M_{\mathbf{e}_{\sigma_k}} \cong M_{\mathbf{e}_{\sigma_k} + (\mathbf{u} - \mathbf{e}_{\sigma_k})} = M_{\mathbf{u}}$$

as desired, where the third isomorphism follows from condition (b') of squarefree P-modules. ☐

The above lemma determines the Krull dimension of a squarefree P-module. Recall that the *(Krull) dimension* $\dim M$ of a finitely generated $\mathbb{Z}$-graded R-module M is the minimal number k such that there is a sequence of homogeneous polynomials $\theta_1, \dots, \theta_k \in R$ of positive degrees such that $\dim_K(M/((\theta_1, \dots, \theta_k)M)) < \infty$.

Corollary 2.4. *If M is a squarefree P-module, then the Krull dimension of M is $\max\{\mathrm{rank}\, \sigma : M_{\mathbf{e}_\sigma} \neq 0\}$.*

Proof. Since the Krull dimension of a $\mathbb{Z}$-graded R-module is equal to the degree of its Hilbert polynomial plus one [BH, Theorem 4.1.3], we have

$$\dim M = \dim \left(\bigoplus_{\sigma \in P} M_{\mathbf{e}_\sigma} \otimes_K K[\langle \sigma \rangle] \right) = \max\{\dim K[\langle \sigma \rangle] : M_{\mathbf{e}_\sigma} \neq 0\}.$$

Then the desired equation follows since $\dim K[\langle \sigma \rangle] = \mathrm{rank}\, \sigma$ by [St3, II Theorem 1.3]. ☐

Hilbert series and flag h-vectors. Squarefree P-modules have a natural $\mathbb{N}^n$-graded structure defined by $\deg x_\sigma = \mathbf{e}_{\mathrm{rank}\, \sigma} \in \mathbb{N}^n$, where $\mathbf{e}_i$ denotes the ith unit vector of $\mathbb{N}^n$. Let M be a squarefree P-module of dimension d. By the above $\mathbb{N}^n$-grading, M is actually $\mathbb{N}^d$-graded since Corollary 2.4 says that $M_{\mathbf{u}} = 0$ for all $\mathbf{u} \in \mathbb{N}^{|\widehat{P}|}$ with $\mathrm{rank}(\max(\mathrm{supp}(\mathbf{u}))) > d$. The ($\mathbb{N}^d$*-graded*) *Hilbert series* of M is the formal power series

$$H_M(t_1, \dots, t_d) = \sum_{\mathbf{v} \in \mathbb{N}^d} (\dim_K M_{\mathbf{v}}) \mathbf{t}^{\mathbf{v}}$$

where $\mathbf{t}^{\mathbf{v}} = t_1^{v_1} \cdots t_d^{v_d}$ for $\mathbf{v} = (v_1, \dots, v_d) \in \mathbb{N}^d$. We use this $\mathbb{N}^d$-grading only when we consider Hilbert series.

For $S \subset [d]$, let $\mathbf{e}_S = \sum_{i \in S} \mathbf{e}_i \in \mathbb{N}^d$ and $\mathbf{t}^S = \prod_{i \in S} t_i$. For a squarefree P-module M of dimension d, we define its *flag f-vector* $(f_S(M) : S \subset [d])$ and its *flag h-vector* $(h_S(M) : S \subset [d])$ by

$$f_S(M) = \dim_K M_{\mathbf{e}_S}$$

and

$$h_S(M) = \sum_{T \subset S} (-1)^{|S|-|T|} f_T(M),$$

where $f_\emptyset(M) = h_\emptyset(M) = \dim_K M_{\mathbf{0}}$. Note that if $M = K[P]$, then $d = n$ and we have $f_S(K[P]) = f_S(P)$ and $h_S(K[P]) = h_S(P)$ for all $S \subset [n]$.

Lemma 2.5. *If M is a squarefree P-module of dimension d, then*

$$H_M(t_1, \dots, t_d) = \frac{\sum_{S\subset[d]} h_S(M)\mathbf{t}^S}{(1-t_1)(1-t_2)\cdots(1-t_d)}.$$

Proof. Since the multiplication $\times x_\tau : M_{\mathbf{u}} \to M_{\mathbf{u}+\mathbf{e}_\tau}$ is bijective if $\tau \in \operatorname{supp}(\mathbf{u})$, we have the decomposition

$$M \cong \bigoplus_{\substack{C\in\Delta_P,\\ \operatorname{rank}(\max(C))\le d}} \Big(M_{(\sum_{\sigma\in C}\mathbf{e}_\sigma)} \otimes_K K[x_\sigma : \sigma\in C]\Big)$$

as $\mathbb{N}^{|\widehat{P}|}$-graded K-vector spaces, where $M_{(\sum_{\sigma\in C}\mathbf{e}_\sigma)} = M_{\mathbf{0}}$ and $K[x_\sigma : \sigma \in C] = K$ if $C = \emptyset$. Then

$$\begin{aligned} H_M(t_1,\dots,t_d) &= \sum_{\substack{C\in\Delta_P,\\ \operatorname{rank}(\max(C))\le d}} \Big(\dim_K M_{(\sum_{\sigma\in C}\mathbf{e}_\sigma)}\Big)\cdot \frac{\prod_{\sigma\in C} t_{\operatorname{rank}\sigma}}{\prod_{\sigma\in C}(1-t_{\operatorname{rank}\sigma})} \\ &= \sum_{S\subset[d]} f_S(M)\cdot \frac{\mathbf{t}^S\cdot\prod_{i\in[d]\setminus S}(1-t_i)}{(1-t_1)(1-t_2)\cdots(1-t_d)} \\ &= \frac{\sum_{S\subset[d]} h_S(M)\mathbf{t}^S}{(1-t_1)(1-t_2)\cdots(1-t_d)}, \end{aligned}$$

as desired. □

For a squarefree P-module M of dimension d, the **ab**-*index* of M is the polynomial $\Psi_M(\mathbf{a},\mathbf{b}) = \sum_{S\subset[d]} h_S(M)w_S$, where w_S is the characteristic monomial of S defined in the introduction. The next formula, which is an analogue of [BE, Equation (3.1)], gives another way to express the flag h-vector of M.

Lemma 2.6. *If M is a squarefree P-module of dimension d, then*

$$\Psi_M(\mathbf{a},\mathbf{b}) = (\dim_K M_{\mathbf{0}})(\mathbf{a}-\mathbf{b})^d + \sum_{\sigma\in\widehat{P}} (\dim_K M_{\mathbf{e}_\sigma})\Psi_{\partial\sigma}(\mathbf{a},\mathbf{b})\cdot\mathbf{b}(\mathbf{a}-\mathbf{b})^{d-\operatorname{rank}\sigma}.$$

Proof. Observe that $K[\langle\sigma\rangle]$ is the polynomial ring over $K[\partial\sigma]$ with the variable x_σ, that is, $K[\langle\sigma\rangle] = K[\partial\sigma][x_\sigma]$. Hence, for any $\sigma\in\widehat{P}$ with $\operatorname{rank}\sigma = r \le d$, we have

$$H_{K[\langle\sigma\rangle]}(t_1,\dots,t_d) = \frac{\sum_{S\subset[r-1]} h_S(\partial\sigma)\mathbf{t}^S}{(1-t_1)(1-t_2)\cdots(1-t_r)}.$$

(Here we identify $H_{K[\langle\sigma\rangle]}(t_1,\dots,t_r)$ and $H_{K[\langle\sigma\rangle]}(t_1,\dots,t_d)$.) Then, by Lemma 2.3, we have

$$\begin{aligned} H_M(t_1,\dots,t_d) &= (\dim_K M_{\mathbf{0}}) + \sum_{\sigma\in\widehat{P}} (\dim_K M_{\mathbf{e}_\sigma}) t_{\operatorname{rank}\sigma}\cdot H_{K[\langle\sigma\rangle]}(t_1,\dots,t_d) \\ &= \frac{\sum_{\sigma\in P}(\dim_K M_{\mathbf{e}_\sigma})t_{\operatorname{rank}\sigma}\Big(\sum_{S\subset[\operatorname{rank}\sigma-1]} h_S(\partial\sigma)\mathbf{t}^S\Big)\prod_{k>\operatorname{rank}\sigma}(1-t_k)}{(1-t_1)(1-t_2)\cdots(1-t_d)}, \end{aligned}$$

where we consider $t_{\mathrm{rank}\,\hat{0}} = 1$. Then, by translating the numerator of the above formula into an **ab**-polynomial, we obtain the desired formula. □

Sheaves on posets and squarefree P-modules. Here we discuss relations between sheaves on P and squarefree P-modules. A *sheaf* $\mathcal{F}$ (of finite K-vector spaces) on P consists of the data

- A finite K-vector space $\mathcal{F}_\sigma$ for each $\sigma \in P$, called the *stalk* of $\mathcal{F}$ at σ.
- Linear maps $\mathrm{res}^\sigma_\tau : \mathcal{F}_\sigma \to \mathcal{F}_\tau$ for all $\sigma > \tau$ in P, called the *restriction maps*, satisfying $\mathrm{res}^\tau_\rho \circ \mathrm{res}^\sigma_\tau = \mathrm{res}^\sigma_\rho$ for all $\sigma > \tau > \rho$ in P.

A *map of sheaves* $\mathcal{F} \to \mathcal{G}$ is a collection of linear maps $\mathcal{F}_\sigma \to \mathcal{G}_\sigma$ commuting with the restriction maps. Note that the formal definition of sheaves is more complicated but it is equivalent to the above one.

Let M be a squarefree P-module. We can construct a sheaf $\mathcal{F}^M$ on P as follows: For $\sigma > \tau$ in P, we define the map $\mathrm{mult}^\sigma_\tau : M_{\mathbf{e}_\tau} \to M_{\mathbf{e}_\sigma}$ by the composition

$$M_{\mathbf{e}_\tau} \xrightarrow{\times x_\sigma} M_{\mathbf{e}_\sigma + \mathbf{e}_\tau} \xrightarrow{(\times x_\tau)^{-1}} M_{\mathbf{e}_\sigma}, \tag{2}$$

where $(\times x_\tau)^{-1}$ is the inverse map of the bijection $\times x_\tau : M_{\mathbf{e}_\sigma} \to M_{\mathbf{e}_\sigma + \mathbf{e}_\tau}$. Then, it is straightforward that these maps satisfy

$$\mathrm{mult}^\sigma_\tau \circ \mathrm{mult}^\tau_\rho = \mathrm{mult}^\sigma_\rho$$

for all $\sigma > \tau > \rho$ in P. We define the sheaf $\mathcal{F}^M$ on P by $\mathcal{F}^M_\sigma = (M_{\mathbf{e}_\sigma})^*$ for all $\sigma \in P$ and $\mathrm{res}^\sigma_\tau = (\mathrm{mult}^\sigma_\tau)^*$ for all $\sigma > \tau$ in P, where $*$ denotes the K-dual.

Conversely, from a sheaf $\mathcal{F}$ on P, we can define the squarefree P-module $M(\mathcal{F})$ as follows: As $\mathbb{N}^{|\widehat{P}|}$-graded K-vector spaces, we define

$$M(\mathcal{F}) = \bigoplus_{\sigma \in P} (\mathcal{F}_\sigma)^* \otimes_K K[\langle \sigma \rangle],$$

where we consider that each element of $(\mathcal{F}_\sigma)^*$ has degree $\mathbf{e}_\sigma$. Then we define the multiplication structure by the following rule. For $\rho \in \widehat{P}$ and $m \otimes f \in (\mathcal{F}_\sigma)^* \otimes_K K[\langle \sigma \rangle]$, we define

$$x_\rho \cdot (m \otimes f) = \begin{cases} (\mathrm{res}^\rho_\sigma)^*(m) \otimes x_\sigma f \quad (\in (\mathcal{F}_\rho)^* \otimes_K K[\langle \rho \rangle]), & \text{if } \rho > \sigma, \\ m \otimes x_\rho f \quad (\in (\mathcal{F}_\sigma)^* \otimes_K K[\langle \sigma \rangle]), & \text{if } \rho \leq \sigma, \\ 0, & \text{otherwise,} \end{cases}$$

where $x_\sigma f \in K[\langle \rho \rangle]$ is the image of f by the injection $K[\langle \sigma \rangle] \xrightarrow{\times x_\sigma} K[\langle \rho \rangle]$.

It is easy to see that $M(\mathcal{F}^N) \cong N$ and $\mathcal{F}^{M(\mathcal{G})} \cong \mathcal{G}$ for any squarefree P-module N and for any sheaf $\mathcal{G}$ on P. Later in Section 4, we will discuss the anti-equivalence between the category of squarefree P-modules and that of sheaves on P, which is given by the correspondences $M \mapsto \mathcal{F}^M$ and $\mathcal{F} \mapsto M(\mathcal{F})$.

3. Homological properties of squarefree P-modules

In this section, we study homological properties of squarefree P-modules when P is a quasi CW-poset. In Sections 3 and 4, we fix a quasi CW-poset $P = \bigcup_{i=0}^n P_i$ of rank n, where $P_i = \{\sigma \in P : \mathrm{rank}\, \sigma = i\}$, and let $R = K[x_\sigma : \sigma \in \widehat{P}]$. We consider the $\mathbb{Z}^{|\widehat{P}|}$-grading of R instead of $\mathbb{N}^{|\widehat{P}|}$-grading to deal with modules having negative graded components.

Augmented oriented chain complexes. We say that an element $\sigma \in P$ *covers* $\tau \in P$ if $\sigma > \tau$ and $\operatorname{rank} \sigma = \operatorname{rank} \tau + 1$. Recall that, since P is a quasi CW-poset, for all $\sigma > \rho$ in P with $\operatorname{rank} \sigma = \operatorname{rank} \rho + 2$, there are exactly two elements τ_1, τ_2 with $\sigma > \tau_i > \rho$. An *incidence function* ε of P is a function $\varepsilon : P \times P \to K$ satisfying the following conditions

(i) $\varepsilon(\sigma, \tau) \neq 0$ if and only if σ covers τ.
(ii) for cover relations $\sigma > \tau_1 > \rho$ and $\sigma > \tau_2 > \rho$ with $\tau_1 \neq \tau_2$, one has

$$\varepsilon(\sigma, \tau_1)\varepsilon(\tau_1, \rho) + \varepsilon(\sigma, \tau_2)\varepsilon(\tau_2, \rho) = 0.$$

For every quasi CW-poset, its incidence function exists and is unique in a certain sense (i.e., in the sense that the augmented oriented chain complex described below is independent of the choice of an incidence function up to isomorphism of complexes). Indeed, for a CW-poset P, an incidence function of P coincides with that of the corresponding regular CW-complex, and the existence and the uniqueness are standard in combinatorial topology (see e.g., [LW, V Theorem 4.2] or [Ma, IV Theorem 7.2]). On the other hand, the corresponding statement for quasi CW-posets is obtained by the same proof since these results for finite regular CW-complexes works even if we allow a closed cell to be the cone of a homology sphere (here a homology sphere means a space which is homeomorphic to a Gorenstein* simplicial complex), and quasi CW-posets are the face posets of such *generalized* regular CW-complexes.

By using an incidence function ε of P, we define the *augmented oriented chain complex* $\mathscr{C}_\bullet^P$ of P as the complex

$$\mathscr{C}_\bullet^P : 0 \longrightarrow \mathscr{C}_{n-1}^P \xrightarrow{\partial} \mathscr{C}_{n-2}^P \xrightarrow{\partial} \cdots \longrightarrow \mathscr{C}_0^P \xrightarrow{\partial} \mathscr{C}_{-1}^P \longrightarrow 0$$

where $\mathscr{C}_i^P = \bigoplus_{\sigma \in P_{i+1}} K \cdot \sigma$ is the K-vector space with basis P_{i+1} and where $\partial(\sigma) = \sum_{\tau \in P_i, \tau < \sigma} \varepsilon(\sigma, \tau)\tau$ for $\sigma \in P_{i+1}$.

Let M be a squarefree P-module. We define the *augmented oriented chain complex* of M (or $\mathcal{F}^M$)

$$\mathscr{C}_\bullet^M : 0 \longrightarrow \mathscr{C}_{n-1}^M \xrightarrow{\partial} \mathscr{C}_{n-2}^M \xrightarrow{\partial} \cdots \longrightarrow \mathscr{C}_0^M \xrightarrow{\partial} \mathscr{C}_{-1}^M \longrightarrow 0$$

by $\mathscr{C}_i^M = \bigoplus_{\sigma \in P_{i+1}} \mathcal{F}_\sigma^M \otimes_K (K \cdot \sigma)$ and $\partial(\mu \otimes \sigma) = \sum_{\tau \in P_i, \tau < \sigma} \operatorname{res}_\tau^\sigma(\mu) \otimes \varepsilon(\sigma, \tau)\tau$ for $\sigma \in P_{i+1}$ and $\mu \in \mathcal{F}_\sigma^M = (M_{\mathbf{e}_\sigma})^*$.

Karu complexes. Augmented oriented chain complexes can be naturally extended to complexes of squarefree P-modules. For a squarefree P-module M, we define the complex

$$\mathcal{L}_\bullet^M : 0 \longrightarrow \mathcal{L}_n^M \xrightarrow{\tilde{\partial}} \mathcal{L}_{n-1}^M \xrightarrow{\tilde{\partial}} \cdots \longrightarrow \mathcal{L}_1^M \xrightarrow{\tilde{\partial}} \mathcal{L}_0^M \longrightarrow 0$$

by

$$\mathcal{L}_i^M = \bigoplus_{\sigma \in P_i} \mathcal{F}_\sigma^M \otimes_K (K[\langle \sigma \rangle] \cdot \sigma)$$

(here we consider that elements of $\mathcal{F}^M_\sigma$ have degree 0) and by

$$\widetilde{\partial}(\mu \otimes f\sigma) = \sum_{\substack{\tau \in P_i \\ \tau < \sigma}} \mathrm{res}^\sigma_\tau(\mu) \otimes \varepsilon(\sigma, \tau)\pi_{\sigma,\tau}(f)\tau$$

for $\mu \otimes f\sigma \in \mathcal{F}^M_\sigma \otimes_K (K[\langle\sigma\rangle]\cdot\sigma)$ with $\operatorname{rank}\sigma = i+1$, where $\pi_{\sigma,\tau}$ is a natural surjection $K[\langle\sigma\rangle] \twoheadrightarrow K[\langle\tau\rangle]$. We call $\mathcal{L}^M_\bullet$ the *Karu complex* of M.

For a $\mathbb{Z}^{|\widehat{P}|}$-graded R-module N and $\mathbf{u} \in \mathbb{Z}^{|\widehat{P}|}$, we write $N(\mathbf{u})$ for the graded module N with grading shifted by $\mathbf{u}$. The Karu complex has the following important property.

Theorem 3.1. *For a squarefree P-module M, one has*

$$H_i(\mathcal{L}^M_\bullet) \cong \mathrm{Ext}^{|\widehat{P}|-i}_R(M, R(-\mathbf{1}))$$

for $i = 0, 1, \ldots, n$, where $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{Z}^{|\widehat{P}|}$.

We will prove the above theorem in the next section since it requires preparation.

Corollary 3.2. *If M is a squarefree P-module, then so is $\mathrm{Ext}^i_R(M, R(-\mathbf{1}))$ for all i.*

Proof. Observe that $\dim M \le n$. Then $\mathrm{Ext}^i_R(M, R(-\mathbf{1})) = 0$ for $i < |\widehat{P}| - n$ by [BH, Corollary 3.5.11]. Since $\mathcal{L}^M_\bullet$ is a complex of squarefree P-modules, its homologies are also squarefree P-modules by Lemma 2.2. So the assertion follows from Theorem 3.1. ☐

Cohen–Macaulay criterion. For $\sigma \in P$, the poset $\mathrm{lk}_P(\sigma) = \{\tau \in P : \tau \ge \sigma\}$ is called the *link of* σ in P and the poset $\mathrm{cost}_P(\sigma) = \{\tau \in P : \tau \not\ge \sigma\}$ is called the *contrastar of* σ in P. Note that $\mathrm{cost}_P(\sigma)$ is an order ideal of P and $\mathrm{lk}_P(\sigma)$ is a quasi CW-poset of rank $\le n - \operatorname{rank}\sigma$. The poset $\mathrm{lk}_P(\sigma)$ is called a star in [Ka], but we call it a link since if P is the face poset of a simplicial complex, then this poset corresponds to a link of a simplicial complex.

Let $\mathcal{F}$ be a sheaf on P. We write $\widetilde{H}_i(\mathcal{F}) = H_i(\mathscr{C}^{\mathcal{F}}_\bullet)$. For $\sigma \in P$, let $\mathrm{lk}_{\mathcal{F}}(\sigma)$ (resp. $\mathrm{cost}_{\mathcal{F}}(\sigma)$) be the sheaf on $\mathrm{lk}_P(\sigma)$ (resp. P) whose stalks and restriction maps are restricted to $\mathrm{lk}_P(\sigma)$ (resp. $\mathrm{cost}_P(\sigma)$). Let M be a squarefree P-module and $\mathcal{F} = \mathcal{F}^M$. Then, by the definition of the Karu complex, it is easy to see that $(\mathcal{L}^M_{\bullet-1})_{\mathbf{e}_\sigma} \cong \mathscr{C}^{\mathcal{F}}_\bullet / (\mathscr{C}^{\mathrm{cost}_{\mathcal{F}}(\sigma)}_\bullet)$ for $\sigma \in P$. Thus

$$H_i((\mathcal{L}^M_\bullet)_{\mathbf{e}_\sigma}) \cong H_{i-1}(\mathscr{C}^{\mathcal{F}}_\bullet / (\mathscr{C}^{\mathrm{cost}_{\mathcal{F}}(\sigma)}_\bullet)) \cong \widetilde{H}_{i+\operatorname{rank}\sigma-1}(\mathrm{lk}_{\mathcal{F}}(\sigma)). \tag{3}$$

Recall that a finitely generated $\mathbb{Z}$-graded R-module M of dimension d is Cohen–Macaulay if and only if $\mathrm{Ext}^i_R(M, R) = 0$ for $i \ne |\widehat{P}| - d$ (see [BH, Corollary 3.5.11]). Then (3) and Theorem 3.1 imply

Theorem 3.3. *Let M be a squarefree P-module of dimension d and $\mathcal{F} = \mathcal{F}^M$. Then M is Cohen–Macaulay if and only if, for any $\sigma \in P$, $\widetilde{H}_i(\mathrm{lk}_{\mathcal{F}}(\sigma)) = 0$ for all $i \ne d - 1 - \operatorname{rank}\sigma$.*

Remark 3.4. Theorems 3.1 and 3.3 are ring-theoretic interpretations of the results in [Ka, EK]. Indeed, the Karu complex of M is the cellular complex of $\mathcal{F}^M \otimes \mathcal{L}$ in [Ka, Section 2.2] and conditions in Theorem 3.3 were used in [EK] as a definition of the Cohen–Macaulay property of sheaves. Also, about Corollary 3.2, the essentially same statement was proved in [EK, Lemma 5.3] for canonical modules. However, we remark that in [Ka] the complex $\mathcal{L}_\bullet^M$ was treated as a complex of $K[\theta_1,\dots,\theta_d]$-module, where $\theta_1,\dots,\theta_d$ is a certain l.s.o.p. of $K[P]$.

Canonical modules. For a $\mathbb{Z}^{|\widehat{P}|}$-graded R-module M of dimension d, the module

$$\Omega(M)=\operatorname{Ext}_R^{|\widehat{P}|-d}(M,R(-\mathbf{1}))$$

is called the *($\mathbb{Z}^{|\widehat{P}|}$-graded) canonical module* of M. By Corollary 3.2, if M is a squarefree P-module then so is $\Omega(M)$. We recall some known properties of canonical modules which are used in the latter sections.

It is well-known in commutative algebra that, if M is a finitely generated $\mathbb{Z}^d$-graded Cohen–Macaulay R-module, then $\Omega(M)$ is Cohen–Macaulay and its $\mathbb{Z}^d$-graded Hilbert series is given by $H_{\Omega(M)}(t_1,\dots,t_d)=(-1)^{\dim M}H_M(\frac{1}{t_1},\dots,\frac{1}{t_d})$ (see [St3, p. 49]). This implies the following duality of flag h-vectors.

Lemma 3.5. *If M is a d-dimensional Cohen–Macaulay squarefree P-module, then $\Omega(M)$ is a d-dimensional Cohen–Macaulay squarefree P-module with $\Psi_{\Omega(M)}(\mathbf{a},\mathbf{b})=\Psi_M(\mathbf{b},\mathbf{a})$.*

Note that the latter condition in the above lemma says $h_S(\Omega(M))=h_{[d]\setminus S}(M)$ for all $S\subset[d]$.

Recall that a *linear system of parameters* (l.s.o.p. for short) of a finitely generated $\mathbb{Z}$-graded R-module M of dimension d is a sequence $\Theta=\theta_1,\dots,\theta_d\in R$ of linear forms such that $\dim_K(M/\Theta M)<\infty$. An l.s.o.p. exists if K is infinite. See [St3, I Lemma 5.2]. For a $\mathbb{Z}$-graded R-module M and for an integer $k\in\mathbb{Z}$, let M_k be the graded component of M of degree k and let $M(k)$ be the graded module M with grading shifted by k. Also, we write M^T for the $\mathbb{Z}$-graded Matlis dual of M [BH, Section 3.6] and $\underline{\Omega}(M)=\operatorname{Ext}_R^{|\widehat{P}|-d}(M,R(-|\widehat{P}|))$ for the $\mathbb{Z}$-graded canonical module of M. Note that, for an $\mathbb{N}^{|\widehat{P}|}$-graded R-module M, $\Omega(M)$ is the same as $\underline{\Omega}(M)$ if we regard it as a $\mathbb{Z}$-graded module. The following fact is more or less well-known, but we give a proof for completeness.

Lemma 3.6. *Let M be a finitely generated $\mathbb{Z}$-graded Cohen–Macaulay R-module of dimension d and Θ an l.s.o.p. of M. Then Θ is an l.s.o.p. of $\underline{\Omega}(M)$ and*

$$(M/(\Theta M))^T\cong(\underline{\Omega}(M)/(\Theta\cdot\underline{\Omega}(M)))(+d).$$

Proof. If $d=0$, the assertion follows from the graded local duality [BH, Theorem 3.6.19(b)]. In fact, if $d=0$ then M equals to its 0th local cohomology module $H^0_\mathfrak{m}(M)$, where $\mathfrak{m}$ is the graded maximal ideal of R, and the local duality says $H^0_\mathfrak{m}(M)^T\cong\underline{\Omega}(M)$. Assume $d\geq 1$ and set $\Theta=\theta_1,\dots,\theta_d$. By the long exact sequence of $\operatorname{Ext}^\bullet_R(-,R(-|\widehat{P}|))$ induced by

$$0\longrightarrow M(-1)\xrightarrow{\times\theta_1}M\longrightarrow M/\theta_1M\longrightarrow 0,$$

we have $\underline{\Omega}(M/\theta_1M)\cong(\underline{\Omega}(M)/\theta_1\underline{\Omega}(M))(+1)$. Repeating this argument, we have

$$\underline{\Omega}(M/\Theta M)\cong\big(\underline{\Omega}(M)/(\Theta\cdot\underline{\Omega}(M))\big)(+d).$$

Since $M/\Theta M$ is a 0-dimensional (Cohen–Macaulay) R-module, we have $(M/\Theta M)^T \cong \underline{\Omega}(M/\Theta M)$. Summing up the above equations, we get the desired statement. □

Skeletons. For a sheaf $\mathcal{F}$ on P, we define its dimension by $\dim \mathcal{F} = \max\{\operatorname{rank} \sigma : \sigma \in P, \mathcal{F}_\sigma \neq 0\}$. Thus $\dim \mathcal{F} = \dim M(\mathcal{F})$. For a sheaf $\mathcal{F}$ on P of dimension d, Karu [Ka] defined its dual sheaf $\mathcal{F}^\vee$ as follows: The stalks of $\mathcal{F}^\vee$ are defined by $\mathcal{F}^\vee_\sigma = H_{d-1}(\mathscr{C}^{\mathcal{F}}_\bullet/\mathscr{C}^{\operatorname{cost}_\mathcal{F}(\sigma)}_\bullet)^*$ and the restriction maps of $\mathcal{F}^\vee$ are the maps induced by the K-dual of the natural surjection

$$\mathscr{C}^{\mathcal{F}}_\bullet/\mathscr{C}^{\operatorname{cost}_\mathcal{F}(\tau)}_\bullet \twoheadrightarrow \mathscr{C}^{\mathcal{F}}_\bullet/\mathscr{C}^{\operatorname{cost}_\mathcal{F}(\sigma)}_\bullet \tag{4}$$

for $\sigma > \tau$. It is not difficult to see that taking the dual sheaf $\mathcal{F}^\vee$ is essentially the same as taking the canonical module, namely,

$$\mathcal{F}^{\Omega(M)} \cong (\mathcal{F}^M)^\vee.$$

Indeed, for a squarefree P-module M of dimension d with $\mathcal{F} = \mathcal{F}^M$, one easily verifies from (3) that

$$(\mathcal{F}^M)^\vee_\sigma = H_{d-1}(\mathscr{C}^{\mathcal{F}}_\bullet/\mathscr{C}^{\operatorname{cost}_\mathcal{F}(\sigma)}_\bullet)^* \cong H_d((\mathcal{L}^M_\bullet)_{\mathbf{e}_\sigma})^* \cong \Omega(M)^*_{\mathbf{e}_\sigma} \cong \mathcal{F}^{\Omega(M)}_\sigma$$

and that the restriction maps of $\mathcal{F}^{\Omega(M)}$ are induced by the surjections (4) since they correspond to the multiplication maps $\operatorname{mult}^\sigma_\tau : \Omega(M)_{\mathbf{e}_\tau} \to \Omega(M)_{\mathbf{e}_\sigma}$ and since, by the identifications $\Omega(M)_{\mathbf{e}_\sigma} \cong H_{d-1}(\mathscr{C}^{\mathcal{F}}_\bullet/\mathscr{C}^{\operatorname{cost}_\mathcal{F}(\sigma)}_\bullet)$ and $\Omega(M)_{\mathbf{e}_\tau} \cong H_{d-1}(\mathscr{C}^{\mathcal{F}}_\bullet/\mathscr{C}^{\operatorname{cost}_\mathcal{F}(\tau)}_\bullet)$, these multiplication maps in $H_d(\mathcal{L}^M_\bullet)$ are induced from (4).

For an integer $k < n$, the poset $P^{(k)} = \{\sigma \in P : \operatorname{rank} \sigma \leq k+1\}$ is called the *k-skeleton* of P. For a sheaf $\mathcal{F}$ on P, we define its *k-skeleton* $\mathcal{F}^{(k)}$ to be the sheaf whose stalks and restriction maps are restricted in $P^{(k)}$. Also, for a squarefree P-module M, we define its k-skeleton $M^{(k)}$ by

$$M^{(k)} = M/(\textstyle\sum_{\sigma \in P \setminus P^{(k)}} M_{\mathbf{e}_\sigma} R) \cong \bigoplus_{\sigma \in P^{(k)}} M_{\mathbf{e}_\sigma} \otimes_K K[\langle \sigma \rangle],$$

where the last isomorphism is an isomorphism as K-vector spaces. Note that $M^{(k)} = M((\mathcal{F}^M)^{(k)})$ and, by the criterion of the Cohen–Macaulay property, $M^{(k)}$ is Cohen–Macaulay if M is Cohen–Macaulay. Karu proved that, for a Cohen–Macaulay sheaf $\mathcal{F}$ on P over $\mathbb{R}$ and for $k < \dim \mathcal{F} - 1$, there is a surjection $(\mathcal{F}^{(k)})^\vee \to \mathcal{F}^{(k)}$ (see [EK, pp. 249–250]). This result of Karu implies the following statement for canonical modules.

Theorem 3.7 (Karu)**.** *Let P be a quasi CW-poset and M a Cohen–Macaulay squarefree P-module of dimension d over $\mathbb{R}$. For $k < d-1$, there is an injection $M^{(k)} \to \Omega(M^{(k)})$.*

Proof. Recall that $M(\mathcal{F}^N) = N$ for any squarefree P-module N and that a surjection $\mathcal{F} \to \mathcal{G}$ between sheaves on P induces an injection $M(\mathcal{G}) \to M(\mathcal{F})$ by the definition of $M(-)$. Observe $(\mathcal{F}^M)^{(k)} = \mathcal{F}^{M^{(k)}}$. Karu's result says that there is a surjection from $((\mathcal{F}^M)^{(k)})^\vee = (\mathcal{F}^{M^{(k)}})^\vee$ to $(\mathcal{F}^M)^{(k)} = \mathcal{F}^{M^{(k)}}$. This implies that there is an injection from $M(\mathcal{F}^{M^{(k)}}) = M^{(k)}$ to $M((\mathcal{F}^{M^{(k)}})^\vee) = \Omega(M^{(k)})$ as desired. □

4. The proof of Theorem 3.1

In this section, we prove Theorem 3.1 as a corollary of a more general result (Theorem 4.11). Since the contents of this section is purely algebraic, readers who are only interested in combinatorics may skip this section. We refer the readers to [Ha] for basics on the theory of derived categories. Before proving the main result, we discuss some properties of squarefree P-modules and Karu complexes.

Squarefree modules. Here we recall squarefree modules over a polynomial ring introduced by the second author [Ya1]. Let $A = K[x_1, \dots, x_m]$ be a polynomial ring with each $\deg x_i = \mathbf{e}_i \in \mathbb{Z}^m$. For $F \subset [m] = \{1, 2, \dots, m\}$, we write $\mathbf{e}_F = \sum_{i \in F} \mathbf{e}_i$ and $K[F] = A/(x_i : i \notin F) \cong K[x_i : i \in F]$.

Definition 4.1. A finitely generated $\mathbb{N}^m$-graded A-module M is called a *squarefree A-module* if it satisfies that, for any $\mathbf{u} = (u_1, \dots, u_m) \in \mathbb{N}^m$ and for any $i \in [m]$ with $u_i > 0$, the multiplication

$$\times x_i : M_{\mathbf{u}} \to M_{\mathbf{u}+\mathbf{e}_i}$$

is bijective.

Let ${}^*\mathrm{Mod}\, A$ be the category of $\mathbb{Z}^m$-graded A-modules and their degree preserving A-homomorphism. Let $\mathrm{Sq}\, A$ be the full subcategory of ${}^*\mathrm{Mod}\, A$ consisting of squarefree A-modules. As shown in [Ya1], $\mathrm{Sq}\, A$ is an abelian subcategory of ${}^*\mathrm{Mod}\, A$. Moreover, $\mathrm{Sq}\, A$ has enough injectives, and any injective object is a finite direct sum of copies of $K[F]$ for various $F \subset [m]$. Below, we recall homological properties of squarefree A-modules studied in [Ya2].

Let ${}^*D_A^\bullet$ be the $\mathbb{Z}^m$-graded dualizing complex of A. Thus ${}^*D_A^\bullet$ is a minimal injective resolution of $A(-\mathbf{1})$, where $\mathbf{1} = \mathbf{e}_{[m]} \in \mathbb{Z}^m$, in ${}^*\mathrm{Mod}\, A$ up to a translation, and has the following description

$${}^*D_A^\bullet : 0 \longrightarrow {}^*D_A^{-m} \xrightarrow{\partial} {}^*D_A^{-m+1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} {}^*D_A^0 \longrightarrow 0$$

with

$${}^*D_A^{-i} = \bigoplus_{\substack{F \subset [m] \\ |F| = i}} {}^*E(K[F]),$$

where ${}^*E(K[F])$ is the injective hull of $K[F]$ in ${}^*\mathrm{Mod}\, A$. If we forget the grading, ${}^*D_A^\bullet$ is quasi-isomorphic to the usual normalized dualizing complex. Note that, since ${}^*D_A^\bullet$ is a $\mathbb{Z}^m$-graded injective resolution of $A(-\mathbf{1})$, we have

$$H^{-i}(\mathrm{Hom}_A^\bullet(M, {}^*D_A^\bullet)) \cong \mathrm{Ext}_A^{m-i}(M, A(-\mathbf{1})) \tag{5}$$

for any finitely generated $\mathbb{Z}^m$-graded A-module M. As shown in [Ya1, Theoerem 2.6], if M is a squarefree A-module, then so is $\mathrm{Ext}_A^i(M, A(-\mathbf{1}))$ for all i. More generally, if $M^\bullet$ is a bounded cochain complex of squarefree A-modules, then $H^i(\mathrm{Hom}_A^\bullet(M^\bullet, {}^*D_A^\bullet))$ is a squarefree A-module for all i (see Section 3 of [Ya4]).

For a $\mathbb{Z}^m$-graded A-module M, $M_{\geq \mathbf{0}}$ denotes the submodule $\bigoplus_{\mathbf{u} \in \mathbb{N}^m} M_{\mathbf{u}}$, and call it the $\mathbb{N}^m$-*graded part* of M. Let $I_A^\bullet = ({}^*D_A^\bullet)_{\geq \mathbf{0}}$. Then $I_A^\bullet$ is quasi-isomorphic to ${}^*D_A^\bullet$ itself, and $I_A^{-i} = \bigoplus_{F \subset [m], |F| = i} K[F]$ since ${}^*E(K[F])_{\geq \mathbf{0}} = K[F]$ (see e.g. [Ya2, p. 48]). Let Δ be a simplicial complex on $[m]$, that is, a collection of subsets of $[m]$

satisfying that $F \in \Delta$ and $G \subset F$ imply $G \in \Delta$ (we assume that the empty set $\emptyset$ is an element of Δ). Consider the subcomplex of $I_A^\bullet$

$$I_\Delta^\bullet : 0 \longrightarrow I_\Delta^{-m} \xrightarrow{\partial} I_\Delta^{-m+1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} I_\Delta^0 \longrightarrow 0$$

with

$$I_\Delta^{-i} = \bigoplus_{\substack{F \in \Delta \\ |F|=i}} K[F].$$

Note that, for $f \in K[F] \subset I_\Delta^{-i}$, one has

$$\partial(f) = \sum_{i \in F} \pm \pi_{F, F\setminus\{i\}}(f) \ \in \bigoplus_{\substack{G \in \Delta \\ |G|=i-1}} K[G] = I_\Delta^{-i+1},$$

where $\pi_{F,F\setminus\{i\}}$ is the natural subjection $K[F] \twoheadrightarrow K[F \setminus \{i\}]$ and where $\pm$ is given by the standard incidence function of the simplicial complex Δ. We say that $M \in \operatorname{Sq} A$ is supported by Δ if $M_{\mathbf{e}_F} = 0$ for all $F \notin \Delta$. The following result was essentially shown in [Ya2].

Lemma 4.2. *Let Δ be a simplicial complex on $[m]$.*

(i) *If M is a squarefree A-module, then for any subset $F \subset [m]$,*

$$[\operatorname{Hom}_A(M, K[F])]_{\geq \mathbf{0}} \cong (M_{\mathbf{e}_F})^* \otimes_K K[F].$$

(ii) *For a bounded cochain complex $M^\bullet$ of squarefree A-modules supported by Δ, $[\operatorname{Hom}_A^\bullet(M^\bullet, I_\Delta^\bullet)]_{\geq \mathbf{0}}$ and $\operatorname{Hom}_A^\bullet(M^\bullet, {}^*D_A^\bullet)$ are isomorphic in the derived category.*

Remark 4.3. In Lemmas 4.2 and 4.8, we consider that elements in $(M_{\mathbf{e}_F})^*$ and $(M_{\mathbf{e}_\sigma})^*$ have degree $\mathbf{0}$.

Proof of Lemma 4.2. (i) By [Ya2, Lemma 3.20] we have

$$[\operatorname{Hom}_A(M, {}^*E(K[F]))]_{\geq \mathbf{0}} \cong (M_{\mathbf{e}_F})^* \otimes_K K[F]$$

for any $F \subset [m]$. However, since ${}^*E(K[F])_{\geq \mathbf{0}} = K[F]$ and ${}^*E(K[F]) \setminus K[F]$ does not concern the $\mathbb{N}^m$-graded part of $\operatorname{Hom}_A(M, {}^*E(K[F]))$, we have

$$[\operatorname{Hom}_A(M, K[F])]_{\geq \mathbf{0}} \cong [\operatorname{Hom}_A(M, {}^*E(K[F]))]_{\geq \mathbf{0}}, \tag{6}$$

which implies the desired statement.

(ii) Since each M^i is supported by Δ, (i) says that $[\operatorname{Hom}_A^\bullet(M^\bullet, I_A^\bullet)]_{\geq \mathbf{0}}$ is equal to $[\operatorname{Hom}_A^\bullet(M^\bullet, I_\Delta^\bullet)]_{\geq \mathbf{0}}$. Also, (6) and the description of ${}^*D_A^\bullet$ imply $[\operatorname{Hom}_A^\bullet(M^\bullet, I_A^\bullet)]_{\geq \mathbf{0}} \cong [\operatorname{Hom}_A^\bullet(M^\bullet, {}^*D_A^\bullet)]_{\geq \mathbf{0}}$. Since homologies of $\operatorname{Hom}_A^\bullet(M^\bullet, {}^*D_A^\bullet)$ are squarefree A-modules by (5) and since squarefree A-modules are $\mathbb{N}^m$-graded, $[\operatorname{Hom}_A^\bullet(M^\bullet, {}^*D_A^\bullet)]_{\geq \mathbf{0}}$ is quasi-isomorphic to $\operatorname{Hom}_A^\bullet(M^\bullet, {}^*D_A^\bullet)$, which implies the desired statement. □

Consider the case when $A = R$. Since squarefree P-modules are squarefree R-modules supported by Δ_P, Lemma 4.2 implies the following fact.

Corollary 4.4. *If M is a squarefree P-module, then we have $\operatorname{Ext}_R^{|\widehat{P}|-i}(M, R(-\mathbf{1})) \cong H^{-i}([\operatorname{Hom}_R^\bullet(M, I_{\Delta_P}^\bullet)]_{\geq \mathbf{0}})$ for all i.*

Note that a squarefree R-module supported by Δ_P is not necessary a squarefree P-module because of condition (b) of squarefree P-modules.

Category of Squarefree P-modules. Here we discuss the category of squarefree P-modules. Let $\mathrm{Sq}_P\,R$ be the full subcategory of $^*\mathrm{Mod}\,R$ consisting of squarefree P-modules. By Lemma 2.2, $\mathrm{Sq}_P\,R$ is an abelian subcategory of $^*\mathrm{Mod}\,R$. From now on, if there is no danger of confusion, C and C' always denote (possibly empty) chains of $\widehat{P}$, equivalently, faces of Δ_P. For a chain C of $\widehat{P}$, we write $K[C] = R/(x_\sigma : \sigma \notin C)$.

Recall the notion of sheaves on P discussed in the previous sections. Let $\mathrm{Sh}\,P$ denote the category of sheaves of finite vector spaces on the poset P and the maps between them.

Proposition 4.5. *We have the category equivalence* $\mathrm{Sq}_P\,R \cong (\mathrm{Sh}\,P)^{\mathsf{op}}$*, where* op *means the opposite category.*

Proof. This (anti)equivalence is given by the constructions $M \mapsto \mathcal{F}^M$ and $\mathcal{F} \mapsto M(\mathcal{F})$ introduced in Section 2. For a morphism $M \to N$ in $\mathrm{Sq}_P\,R$, let $f_\sigma : M_{\mathbf{e}_\sigma} \to N_{\mathbf{e}_\sigma}$ be the restriction of f to the degree $\mathbf{e}_\sigma$ part. Then the family of K-linear maps $\{(f_\sigma)^*\}_{\sigma\in P}$ gives a morphism $\mathcal{F}^N \to \mathcal{F}^M$ in $\mathrm{Sh}\,P$. It is easy to see that this correspondence gives a contravariant functor $\mathrm{Sq}_P\,R \to \mathrm{Sh}\,P$. Similarly, we can construct a morphism $M(\mathcal{G}) \to M(\mathcal{F})$ in $\mathrm{Sq}_P\,R$ from a morphism $\mathcal{F} \to \mathcal{G}$ in $\mathrm{Sh}\,P$, which gives a contravariant functor $\mathrm{Sh}\,P \to \mathrm{Sq}_P\,R$. Then, since $M(\mathcal{F}^M) \cong M$ and $\mathcal{F}^{M(\mathcal{F})} \cong \mathcal{F}$, we have $\mathrm{Sq}_P\,R \cong (\mathrm{Sh}\,P)^{\mathsf{op}}$. □

Remark 4.6. In [Ya3], a sheaf on a finite poset is defined in the opposite manner. More precisely, the restriction maps of a sheaf $\mathcal{F}$ in [Ya3] are K-linear maps $\mathcal{F}_\sigma \to \mathcal{F}_\tau$ for $\sigma < \tau$. If we use this convention, the category of sheaves on P is directly equivalent to $\mathrm{Sq}_P\,R$, and we do not have to take the opposite category. However, we follow the convention of [Ka, EK] in this paper.

Corollary 4.7. *The category* $\mathrm{Sq}_P\,R$ *has enough injectives, and any injective object is a finite direct sum of copies of* $K[\langle\sigma\rangle]$ *for various* $\sigma \in P$.

Proof. It is well-known that $\mathrm{Sh}\,P$ is an abelian category with enough projectives and injectives, and an indecomposable projective object is of the form $\mathcal{P}(\sigma)$ for some $\sigma \in P$, where

$$\mathcal{P}(\sigma)_\tau = \begin{cases} K, & \text{if } \tau \le \sigma, \\ 0, & \text{otherwise,} \end{cases}$$

and all the restriction map $\mathrm{res}^\tau_\rho : \mathcal{P}(\sigma)_\tau \to \mathcal{P}(\sigma)_\rho$ are injective for all $\tau, \rho \in P$ with $\tau > \rho$. See, for example, [Ya3] and references cited therein (the reader should be careful with the point that the convention on sheaves in [Ya3] is "opposite" to ours as mentioned in Remark 4.6).

Since $\mathrm{Sq}_P\,R \cong (\mathrm{Sh}\,P)^{\mathsf{op}}$, $\mathrm{Sq}_P\,R$ also has enough projectives and injectives, and injective objects in $\mathrm{Sq}_P\,R$ correspond to projective objects in $\mathrm{Sh}\,P$. Moreover, it is easy to check that $M(\mathcal{P}(\sigma)) \cong K[\langle\sigma\rangle]$. So we are done. □

Another description of a Karu complex. Next, we show that the Karu complex $\mathcal{L}^M_\bullet$ can be described in a way similar to Lemma 4.2. We define the complex $J^\bullet_P$ by $J^\bullet_P = \mathcal{L}^{K[P]}_{-\bullet}$. Thus, $J^\bullet_P$ is the complex of squarefree P-modules of the form

$$J^\bullet_P : 0 \longrightarrow J^{-n}_P \xrightarrow{\widetilde{\partial}} J^{-n+1}_P \xrightarrow{\widetilde{\partial}} \cdots \xrightarrow{\widetilde{\partial}} J^0_P \longrightarrow 0$$

with $J_P^{-i} = \bigoplus_{\sigma \in P_i} K[\langle\sigma\rangle] \cdot \sigma$. For simplicity, we write $J_P^{-i} = \bigoplus_{\sigma \in P_i} K[\langle\sigma\rangle]$ in the rest of this section. Then, for $f \in K[\langle\sigma\rangle]$ with $\operatorname{rank}\sigma = i$,

$$\widetilde{\partial}(f) = \sum_{\substack{\tau \in P \\ \tau < \sigma}} \varepsilon(\sigma,\tau) \cdot \pi_{\sigma,\tau}(f) \ \in J_P^{-i+1} = \bigoplus_{\tau \in P_{i-1}} K[\langle\tau\rangle],$$

where ε is the fixed incidence function and $\pi_{\sigma,\tau}$ is the natural subjection $K[\langle\sigma\rangle] \twoheadrightarrow K[\langle\tau\rangle]$.

Lemma 4.8. *Let $M \in \operatorname{Sq}_P R$ and $\sigma \in P$ with $\operatorname{rank}\sigma = r$.*

(i) $[\operatorname{Hom}_R(M, K[\langle\sigma\rangle])]_{\geq \mathbf{0}} \cong (M_{\mathbf{e}_\sigma})^* \otimes_K K[\langle\sigma\rangle]$.

(ii) $[\operatorname{Hom}^\bullet_R(M, J_P^\bullet)]_{\geq \mathbf{0}} \cong \mathcal{L}^M_{-\bullet}$.

(iii) $H_k(\mathcal{L}_\bullet^{K[\langle\sigma\rangle]}) = 0$ *for* $k \neq r$ *and* $H_r(\mathcal{L}_\bullet^{K[\langle\sigma\rangle]}) \cong x_\sigma K[\langle\sigma\rangle]$.

Proof. (i) We may assume that $\sigma \neq \hat{0}$. Since $\partial\sigma$ is Gorenstein*, $\Omega(K[\partial\sigma]) \cong K[\partial\sigma]$. Then, since $K[\langle\sigma\rangle] = K[\partial\sigma][x_\sigma]$, the canonical module $\Omega(K[\langle\sigma\rangle]) \cong K[\langle\sigma\rangle](-\mathbf{e}_\sigma)$ is isomorphic to $x_\sigma K[\langle\sigma\rangle]$ the ideal of $K[\langle\sigma\rangle]$ generated by x_σ. Let

$$0 \longrightarrow K[\langle\sigma\rangle] \longrightarrow I^0 \stackrel{f}{\longrightarrow} I^1$$

be the first step of the minimal injective resolution of $K[\langle\sigma\rangle]$ in the category $\operatorname{Sq} R$. By [Ya4, Proposition 3.5], each I^i is the direct sum of $\dim_K \Omega(K[\langle\sigma\rangle])_{\mathbf{e}_F}$ copies of $R/(x_\sigma : \sigma \notin F)$ for $F \subset \widehat{P}$ with $|F| = r - i$. Then, since $\Omega(K[\langle\sigma\rangle]) \cong x_\sigma K[\langle\sigma\rangle]$,

$$I^0 \cong \bigoplus_{\substack{\max C = \sigma \\ |C| = r}} K[C] \qquad \text{and} \qquad I^1 \cong \bigoplus_{\substack{\max C' = \sigma \\ |C'| = r-1}} K[C'].$$

Observe

$$\operatorname{Hom}_R(M, K[C])_{\geq \mathbf{0}} \cong (M_{\sum_{\sigma \in C} \mathbf{e}_\sigma})^* \otimes_K K[C] \cong (M_{\mathbf{e}_{\max C}})^* \otimes_K K[C]$$

by Lemma 4.2 and condition (b) of squarefree P-module. Since $\operatorname{Hom}_R(M, K[\langle\sigma\rangle])$ is the kernel of

$$f_* : \operatorname{Hom}_R\left(M, I^0\right) \longrightarrow \operatorname{Hom}_R\left(M, I^1\right),$$

$[\operatorname{Hom}_R(M, K[\langle\sigma\rangle])]_{\geq \mathbf{0}}$ is isomorphic to the kernel of

$$(M_\sigma)^* \otimes_K f : \bigoplus_{\substack{\max C = \sigma \\ |C| = r}} (M_{\mathbf{e}_\sigma})^* \otimes_K K[C] \longrightarrow \bigoplus_{\substack{\max C' = \sigma \\ |C'| = r-1}} (M_{\mathbf{e}_\sigma})^* \otimes_K K[C'].$$

Then the statement follows since $\ker((M_{\mathbf{e}_\sigma})^* \otimes_K f) \cong (M_{\mathbf{e}_\sigma})^* \otimes_K \ker f$ and since $\ker f \cong K[\langle\sigma\rangle]$.

(ii) is an immediate consequence of (i). We prove (iii). Recall that by (3) in Section 3, one has

$$H_k((\mathcal{L}_\bullet^{K[\langle\sigma\rangle]})_{\mathbf{e}_\tau}) \cong \widetilde{H}_{k-1-\operatorname{rank}\tau}(\operatorname{lk}_{\langle\sigma\rangle}(\tau))$$

for all $\tau \leq \sigma$. Since $\operatorname{lk}_{\langle\sigma\rangle}(\tau)$ has the maximal element, its order complex is a cone unless $\sigma = \tau$. Thus the homologies of $\operatorname{lk}_{\langle\sigma\rangle}(\tau)$ are zero except for the case when $k = r$ and $\tau = \sigma$, which implies the first assertion. Also, since $\mathcal{L}_r^{K[\langle\sigma\rangle]} \cong K[\langle\sigma\rangle]$, $H_r(\mathcal{L}_\bullet^{K[\langle\sigma\rangle]})$ is isomorphic to an ideal of $K[\langle\sigma\rangle]$. However, if an ideal in $K[\langle\sigma\rangle]$ is a squarefree P-module then it must be generated by variables. Thus the above fact on homologies of $\mathrm{lk}_{\langle\sigma\rangle}(\tau)$ implies that $H_r(\mathcal{L}_\bullet^{K[\langle\sigma\rangle]})$ is isomorphic to the ideal of $K[\langle\sigma\rangle]$ generated by x_σ. $\square$

We also note the following fact which will be obvious to the specialists.

Lemma 4.9. $[\mathrm{Hom}^\bullet_R(-, I^\bullet_{\Delta_P})]_{\geq \mathbf{0}}$ *and* $[\mathrm{Hom}^\bullet_R(-, J^\bullet_P)]_{\geq \mathbf{0}}$ *give contravariant functors from the bounded derived category* $\mathsf{D}^b(\mathrm{Sq}_P\, R)$ *to* $\mathsf{D}^b({}^*\mathrm{Mod}\, R)$.

Proof. Since $\mathrm{Hom}^\bullet_R(-, {}^*D^\bullet_R)$ is a contravariant functor in $\mathsf{D}^b({}^*\mathrm{Mod}\, R)$, the statement for $I^\bullet_{\Delta_P}$ follows from Lemma 4.2(ii). We consider $J^\bullet_P$. Let $M^\bullet$ be a bounded cochain complex of squarefree P-modules which is acyclic (i.e., $H^i(M^\bullet) = 0$ for all i). What we must prove is that $[\mathrm{Hom}^\bullet_R(M^\bullet, J^\bullet_P)]_{\geq \mathbf{0}}$ is also acyclic. By Lemma 4.8(i), $[\mathrm{Hom}^\bullet_R(M^\bullet, K[\langle\sigma\rangle])]_{\geq \mathbf{0}}$ is acyclic for all $\sigma \in P$. Recall that each J^i_P is a finite direct sum of copies of $K[\langle\sigma\rangle]$. Hence, by the usual double complex argument, we can show that $[\mathrm{Hom}^\bullet_R(M^\bullet, J^\bullet_P)]_{\geq \mathbf{0}}$ is acyclic. $\square$

Main result. We will construct the chain map

$$\tilde{\iota} : J^\bullet_P \to I^\bullet_{\Delta_P}$$

and prove that this chain map induces a quasi-isomorphism from $[\mathrm{Hom}^\bullet_R(M^\bullet, J^\bullet_P)]_{\geq \mathbf{0}}$ to $[\mathrm{Hom}^\bullet_R(M^\bullet, I^\bullet_{\Delta_P})]_{\geq \mathbf{0}}$ for any bounded cochain complex $M^\bullet$ of squarefree P-modules.

Construction 4.10. We construct the map $\tilde{\iota}$. We write $\partial^{-i} : I^{-i}_{\Delta_P} \to I^{-i+1}_{\Delta_P}$ for the boundaries of $I^\bullet_{\Delta_P}$. Fix an incidence function ε of P. Take $\sigma \in P$ with $\mathrm{rank}\, \sigma = r$. The complex $I^\bullet_\sigma = [\mathrm{Hom}^\bullet_R(K[\langle\sigma\rangle], I^\bullet_{\Delta_P})]_{\geq \mathbf{0}}$ can be seen as a subcomplex of $I^\bullet_{\Delta_P}$ with $I^{-i}_\sigma = \bigoplus_{\max C \leq \sigma, |C| = i} K[C]$ by Lemma 4.2(i). The "tail" $0 \to I^{-r}_\sigma \to I^{-r+1}_\sigma$ of the complex $I^\bullet_\sigma$ is of the form

$$0 \longrightarrow \bigoplus_{\substack{\max C = \sigma \\ |C| = r}} K[C] \xrightarrow{\partial^{-r}} \bigoplus_{\substack{\max C' \leq \sigma \\ |C'| = r-1}} K[C'].$$

Since $\ker(\partial^{-r}) \cong \Omega(K[\langle\sigma\rangle])$ by Corollary 5.3, as we saw in the proof of Lemma 4.8(i), $\ker(\partial^{-r})$ is isomorphic to $x_\sigma K[\langle\sigma\rangle]$. Then, by the injectivity of $K[C]$'s in $\mathrm{Sq}\, R$, we have an injection

$$\iota_\sigma : K[\langle\sigma\rangle] \longrightarrow \bigoplus_{\substack{\max C = \sigma \\ |C| = r}} K[C]$$

satisfying $\partial^{-r} \circ \iota_\sigma(x_\sigma) = 0$ by extending the injection

$$x_\sigma K[\langle\sigma\rangle] \cong \ker(\partial^{-r}) \hookrightarrow \bigoplus_{\substack{\max C = \sigma \\ |C| = r}} K[C].$$

Note that ι_σ is unique up to constant multiplications. More precisely, if an injective homomorphism $\iota' : K[\langle\sigma\rangle] \longrightarrow I^{-r}_\sigma$ satisfies $\partial^{-r} \circ \iota'(x_\sigma) = 0$, then we have $\iota' = a \cdot \iota_\sigma$ for some $a \in K \setminus \{0\}$. This is because ι' only depends on the choice of $\iota'(1)$ but, by the injectivity of the multiplication by x_σ in $I^{-r}_\sigma \cong \bigoplus_{\max C = \sigma,\ |C| = r} K[C]$, it actually only depends on the choice of $\iota'(x_\sigma) \in \ker(\partial^{-r})_{\mathbf{e}_\sigma} \cong K$. Also, by the injectivity of ι_σ, one has

$$\operatorname{Im}(\partial^{-r} \circ \iota_\sigma) \cong K[\langle\sigma\rangle]/(x_\sigma K[\langle\sigma\rangle]). \tag{7}$$

We claim that, by appropriate choices of ι_σ, the maps $\{\iota_\sigma\}_{\sigma\in P}$ induces a chain map from $J_P^\bullet$ to $I_{\Delta_P}^\bullet$.

Fix $\{\iota_\sigma\}_{\sigma\in P}$. Let L_σ be the 1-dimensional K-vector space spanned by $\iota_\sigma(1)$ and $L^{-i} = \bigoplus_{\sigma\in P_i} L_\sigma$. We first prove that $(L^\bullet, \partial)$ is a complex. Since $x_\sigma \partial^{-r}(\iota_\sigma(1)) = \partial^{-r}(\iota_\sigma(x_\sigma)) = 0$, where $r = \operatorname{rank}\sigma$, and since the multiplication by x_σ in $K[C]$ with $\sigma \in C$ is injective,

$$\partial^{-r}(\iota_\sigma(1)) \in \bigoplus_{\substack{\max C<\sigma \\ |C|=r-1}} K[C] = \bigoplus_{\sigma \text{ covers } \tau} I_\tau^{-r+1}.$$

Let $\partial^{-r}(\iota_\sigma(1)) = \sum f_\tau$ with $f_\tau \in I_\tau^{-r+1}$. Since $(I_\rho^{-r+1})_{\mathbf{e}_\tau} = 0$ for all $\tau, \rho \in P_{r-1}$ with $\tau \neq \rho$, $\partial^{-r}(\iota_\sigma(x_\tau)) = x_\tau f_\tau$. Since $\partial^{-r+1} \circ \partial^{-r} = 0$, $x_\tau f_\tau$ is contained in the kernel of the map $\partial^{-r+1} : I_\tau^{-r+1} \to I_\tau^{-r+2}$. Moreover, since $\partial^{-r}(\iota_\sigma(x_\tau)) \neq 0$ by (7), $x_\tau f_\tau \neq 0$. Then, by the injectivity of the multiplication by x_τ in I_τ^{-r+1} and by the uniqueness of the map ι_τ, there is an $a_{\sigma,\tau} \in K \setminus \{0\}$ such that $f_\tau = a_{\sigma,\tau} \cdot \iota_\tau(1)$. This implies that

$$\partial^{-r}(\iota_\sigma(1)) = \sum_{\sigma \text{ covers } \tau} a_{\sigma,\tau} \cdot \iota_\tau(1) \in L^{-r+1}, \tag{8}$$

and therefore $L^\bullet$ is a complex.

Observe that $L^\bullet$ is a complex of K-vector spaces with basis $\{\iota_\sigma(1)\}_{\sigma\in P}$ and that, for every $\sigma > \rho$ in P with $\operatorname{rank}\sigma = \operatorname{rank}\rho + 2$, the set $\{\tau \in P : \sigma > \tau > \rho\}$ contains exactly two elements. Since all the coefficients $a_{\sigma,\tau}$ are non-zero in (8), the numbers $\{a_{\sigma,\tau}\}$ give an incidence function on P (in other words, $L^\bullet$ is the augmented oriented chain complex of P). Then the uniqueness of an incidence function of P says that, by replacing ι_σ with its scalar multiple if necessary, we may assume that the equation

$$\partial^{-r} \circ \iota_\sigma(1) = \sum_{\sigma \text{ covers } \tau} \varepsilon(\sigma,\tau) \cdot \iota_\tau(1) \tag{9}$$

holds for all $\sigma \in P$. For each i with $0 \le i \le r$, define the map $\tilde{\iota} : J_P^\bullet \longrightarrow I_{\Delta_P}^\bullet$ by $\tilde{\iota}^{-i} = \sum_{\sigma\in P_i} \iota_\sigma$. Clearly, (9) says that $\tilde{\iota}$ is a chain map.

Now we are in the position to prove the main result of this section.

Theorem 4.11. *For a bounded cochain complex $M^\bullet$ of squarefree P-modules, the complexes $[\operatorname{Hom}_R^\bullet(M^\bullet, J_P^\bullet)]_{\ge \mathbf{0}}$ and $[\operatorname{Hom}_R^\bullet(M^\bullet, I_{\Delta_P}^\bullet)]_{\ge \mathbf{0}}$ are isomorphic in the bounded derived category $\mathsf{D}^b({}^*\mathrm{Mod}\, R)$.*

Proof. By Lemma 4.9, $[\operatorname{Hom}_R^\bullet(-, J_P^\bullet)]_{\ge \mathbf{0}}$ and $[\operatorname{Hom}_R^\bullet(-, I_{\Delta_P}^\bullet)]_{\ge \mathbf{0}}$ are contravariant functors from $\mathsf{D}^b(\mathrm{Sq}_P\, R)$ to $\mathsf{D}^b({}^*\mathrm{Mod}\, R)$. Consider the chain map $\tilde{\iota} : J_P^\bullet \longrightarrow I_{\Delta_P}^\bullet$. Taking the $\mathbb{N}^{|\widehat{P}|}$-graded part of $\tilde{\iota}_* : \operatorname{Hom}_R(M^\bullet, J_P^\bullet) \to \operatorname{Hom}_R^\bullet(M^\bullet, I_{\Delta_P}^\bullet)$, we have the chain map

$$[\operatorname{Hom}_R^\bullet(M^\bullet, J_P^\bullet)]_{\ge \mathbf{0}} \longrightarrow [\operatorname{Hom}_R^\bullet(M^\bullet, I_P^\bullet)]_{\ge \mathbf{0}}.$$

This gives a natural transform $\eta : [\mathrm{Hom}^\bullet_R(-, J^\bullet_P)]_{\geq \mathbf{0}} \to [\mathrm{Hom}^\bullet_R(-, I^\bullet_{\Delta_P})]_{\geq \mathbf{0}}$. By the construction of the chain map $\tilde{\iota}$, it follows that $\eta(K[\langle\sigma\rangle])$ is quasi-isomorphism for all $\sigma \in P$. In fact, if $\operatorname{rank}\sigma = r$, then, since $K[\langle\sigma\rangle]$ is Cohen–Macaulay, both $[\mathrm{Hom}^\bullet_R(K[\langle\sigma\rangle], J^\bullet_P)]_{\geq \mathbf{0}}$ and $[\mathrm{Hom}^\bullet_R(K[\langle\sigma\rangle], I^\bullet_{\Delta_P})]_{\geq \mathbf{0}}$ are exact except at the $(-r)$-th cohomology, which is isomorphic to the ideal $x_\sigma K[\langle\sigma\rangle]$ by Corollary 4.4 and Lemma 4.8. Also, the map between $(-r)$-th term of the complexes coincides with the map ι_σ that sends $x_\sigma K[\langle\sigma\rangle]$ to the kernel of $\partial : I^{-r}_\sigma \to I^{-r+1}_\sigma$ which is isomorphic to $x_\sigma K[\langle\sigma\rangle]$. It means that $\eta(-)$ is quasi-isomorphism for any injective object in $\mathrm{Sq}_P\, R$. Hence applying [Ha, Proposition 7.1], we see that η is a natural isomorphism. □

By Corollary 4.4 and Lemma 4.8(ii), Theorem 3.1 is the special case of Theorem 4.11 when $M^\bullet$ is a single module. Indeed, for a squarefree P-module M, we have isomorphisms

$$\mathrm{Ext}^{|\widehat{P}|-i}_R(M, R(-\mathbf{1})) \cong H^{-i}([\mathrm{Hom}^\bullet_R(M, I^\bullet_{\Delta_P})]_{\geq \mathbf{0}}) \cong H^{-i}([\mathrm{Hom}^\bullet_R(M, J^\bullet_P)]_{\geq \mathbf{0}}) \cong H_i(\mathcal{L}^M_\bullet).$$

Corollary 4.12. *The chain map $\tilde{\iota} : J^\bullet_P \to I^\bullet_{\Delta_P}$ defined in Construction 4.10 is a quasi-isomorphism.*

Proof. We use the notation in the proof of Theorem 4.11. Since $[\mathrm{Hom}^\bullet_R(K[P], J^\bullet_P)]_{\geq \mathbf{0}} = J^\bullet_P$, $[\mathrm{Hom}^\bullet_R(K[P], I^\bullet_{\Delta_P})]_{\geq \mathbf{0}} = I^\bullet_{\Delta_P}$ and $\eta(K[P]) = \tilde{\iota}$, the assertion follows from the fact that η is a natural isomorphism. □

5. Extended **cd**-indices

In this section, we consider an extension of the **cd**-index to quasi CW-posets. Throughout this section, we assume that P is a quasi CW-poset. Recall that $\mathbb{Z}\langle \mathbf{c}, \mathbf{d}\rangle$ denotes the non-commutative polynomial ring with coefficients in $\mathbb{Z}$ with the variables $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$.

We say that a squarefree P-module M of dimension d has the *symmetric flag h-vector* if $h_S(M) = h_{[d]\setminus S}(M)$ for all $S \subset [d]$, equivalently, $\Psi_M(\mathbf{a}, \mathbf{b}) = \Psi_M(\mathbf{b}, \mathbf{a})$. Note that, if P is Gorenstein*, then $K[P]$ has the symmetric flag h-vector (see [St4, Corollary 3.16.6]).

Lemma 5.1. *If M is a squarefree P-module of dimension d, then there are unique **cd**-polynomials $\Phi, \Upsilon \in \mathbb{Z}\langle \mathbf{c}, \mathbf{d}\rangle$ of degrees d and $d-1$ such that*

$$\Psi_M(\mathbf{a}, \mathbf{b}) = \Phi + \Upsilon \mathbf{b} \tag{10}$$

Moreover, if M has the symmetric flag h-vector then $\Psi_M(\mathbf{a}, \mathbf{b}) = \Phi$.

Proof. We first prove the existence. By Lemma 2.6, we have

$$\Psi_M(\mathbf{a}, \mathbf{b}) = (\dim_K M_{\mathbf{0}}) \cdot (\mathbf{a} - \mathbf{b})^d + \sum_{\sigma \in \widehat{P}} (\dim_K M_{\mathbf{e}_\sigma}) \Psi_{\partial\sigma}(\mathbf{a}, \mathbf{b}) \cdot \mathbf{b}(\mathbf{a} - \mathbf{b})^{d - \operatorname{rank}\sigma}.$$

Observe $(\mathbf{a} - \mathbf{b})^d = (\mathbf{c} - 2\mathbf{b})(\mathbf{a} - \mathbf{b})^{d-1}$ and each $\Psi_{\partial\sigma}(\mathbf{a}, \mathbf{b}) \in \mathbb{Z}\langle \mathbf{c}, \mathbf{d}\rangle$. Then, to prove the statement, it is enough to prove that, for any $\Phi, \Upsilon \in \mathbb{Z}\langle \mathbf{c}, \mathbf{d}\rangle$, there are $\Phi', \Upsilon' \in \mathbb{Z}\langle \mathbf{c}, \mathbf{d}\rangle$ such that $(\Phi + \Upsilon\mathbf{b})(\mathbf{a} - \mathbf{b}) = \Phi' + \Upsilon'\mathbf{b}$. Indeed the following computation proves the desired statement.

$$(\Phi + \Upsilon\mathbf{b})(\mathbf{a} - \mathbf{b}) = \Phi \cdot (\mathbf{a} - \mathbf{b}) + \Upsilon \cdot (\mathbf{ba} - \mathbf{b}^2) = \Phi \cdot (\mathbf{c} - 2\mathbf{b}) + \Upsilon \cdot (\mathbf{d} - \mathbf{cb}).$$

Next, the uniqueness of the expression (10) follows since if $\Psi_M(\mathbf{a},\mathbf{b}) = \Phi + \Upsilon\mathbf{b}$, then we have $\Psi_M(\mathbf{a},\mathbf{b}) - \Psi_M(\mathbf{b},\mathbf{a}) = \Upsilon(\mathbf{b}-\mathbf{a})$, which says that $\Psi_M(\mathbf{a},\mathbf{b})$ determines Υ. Finally, if M has the symmetric flag h-vector and $\Psi_M(\mathbf{a},\mathbf{b}) = \Phi + \Upsilon\mathbf{b}$, then one obtains $\Phi + \Upsilon\mathbf{b} = \Phi + \Upsilon\mathbf{a}$, which implies $\Upsilon = 0$. □

We call (10) the **b**-*expression* of $\Psi_M(\mathbf{a},\mathbf{b})$. By substituting $\mathbf{b} = \mathbf{c} - \mathbf{a}$ to (10), one obtains a similar expression

$$\Psi_M(\mathbf{a},\mathbf{b}) = \Phi' + \Upsilon'\mathbf{a},$$

where $\Phi' = \Phi + \Upsilon\mathbf{c}$ and $\Upsilon' = -\Upsilon$. We call the above expression the **a**-*expression* of $\Psi_M(\mathbf{a},\mathbf{b})$. These expressions are not always non-negative. We discuss their non-negativity later in Corollary 5.5.

Recall that F_n denotes the nth Fibonacci number defined by $F_1 = F_2 = 1$ and $F_{k+2} = F_{k+1} + F_k$. Since **cd**-polynomials of degree n have at most F_{n+1} non-zero coefficients, the **a**-expression gives a way to express flag h-vectors of CW-posets of rank n by $F_{n+1} + F_n = F_{n+2}$ integers. We prove that the **a**-expression gives an efficient way to express the flag h-vectors in the sense that it incorporates all linear equations satisfied by the flag h-vectors of all quasi CW-posets. Let $\mathcal{H}_n \subset \mathbb{Z}^{2^n}$ be the set of all flag h-vectors of quasi CW-posets of rank n and $\mathcal{HP}_n \subset \mathcal{H}_n$ the set of all flag h-vectors of the face posets of the polyhedral complexes of dimension $n-1$. Let $\mathbb{R}\mathcal{H}_n$ (resp. $\mathbb{R}\mathcal{HP}_n$) be the $\mathbb{R}$-linear space spanned by $\mathcal{H}_n$ (resp. $\mathcal{HP}_n$). The next result shows that the existence of the **a**-expression describes all linear equations satisfied by the flag h-vectors of all quasi CW-posets (or all polyhedral complexes).

Proposition 5.2. $\dim_{\mathbb{R}} \mathbb{R}\mathcal{H}_n = \dim_{\mathbb{R}} \mathbb{R}\mathcal{HP}_n = F_{n+2}$.

Proof. Since the number of **cd**-monomials of degree n is F_{n+1}, Lemma 5.1 says that $\dim \mathbb{R}\mathcal{H}_n \leq F_{n+2}$. Thus, to prove the statement, it is enough to find F_{n+2} polyhedral complexes of dimension $n-1$ whose **ab**-indices are linearly independent. Note that to prove the linear independence of the **ab**-indices it is enough to prove the linear independence of their **a**-expressions.

For a convex n-polytope Q, we write Ψ_Q and $\Psi_{\partial Q}$ for the **ab**-indices of the face posets of Q and ∂Q respectively. Since $f_S(Q) = f_{S\cup\{n\}}(Q) = f_S(\partial Q)$ for all $S \subset [n-1]$, we have $h_S(Q) = h_S(\partial Q)$ and $h_{S\cup\{n\}}(Q) = 0$ for all $S \subset [n-1]$. Thus $\Psi_Q = \Psi_{\partial Q}\cdot\mathbf{a}$. By [BB, Theorem 2.6], there are n-polytopes $Q_1,\dots,Q_{F_{n+1}}$ and $(n-1)$-polytopes $Q'_1,\dots,Q'_{F_n}$ such that $\Psi_{\partial Q_1},\dots,\Psi_{\partial Q_{F_{n+1}}} \in \mathbb{Z}\langle\mathbf{c},\mathbf{d}\rangle$ are linearly independent polynomials of degree n and $\Psi_{\partial Q'_1},\dots,\Psi_{\partial Q'_{F_n}} \in \mathbb{Z}\langle\mathbf{c},\mathbf{d}\rangle$ are linearly independent polynomials of degree $n-1$. Then, since the **a**-expressions of the polynomials

$$\Psi_{\partial Q_1},\dots,\Psi_{\partial Q_{F_{n+1}}}, \Psi_{Q'_1} = \Psi_{\partial Q'_1}\cdot\mathbf{a},\dots,\Psi_{Q'_{F_n}} = \Psi_{\partial Q'_{F_n}}\cdot\mathbf{a}$$

are linearly independent, we obtain the desired statement. □

It is possible to find linear equations that determine $\mathbb{R}\mathcal{H}_n$ in the same way as in the proof of [BB, Theorem 2.1]. This will give another proof of Proposition 5.2.

Next, we prove Theorem 1.1. Before proving it, we translate Karu's proof of the non-negativity of the **cd**-indices of Gorenstein* posets in the language of commutative algebra. For a Cohen–Macaulay squarefree P-module M such that there is an injection $\phi : M \to \Omega(M)$, we write $\Omega(M)/M = \Omega(M)/\phi(M)$ to simplify the notation. The following statement is due to Karu [Ka, Lemma 4.7].

Lemma 5.3 (Karu)**.** *Let M be a Cohen–Macaulay squarefree P-module of dimension d such that there is an injection $M \to \Omega(M)$ and let $\Psi_M(\mathbf{a},\mathbf{b}) = \Phi + \Upsilon\mathbf{b}$ be the $\mathbf{b}$-expression of $\Psi_M(\mathbf{a},\mathbf{b})$. Then $\Omega(M)/M$ is a Cohen–Macaulay squarefree P-module with $\Psi_{\Omega(M)/M}(\mathbf{a},\mathbf{b}) = \Upsilon$.*

Proof. The Cohen–Macaulay property is standard in commutative algebra. Indeed, since $\Omega(M)$ and M have the same dimension and the multiplicity (see [BH, Proposition 4.1.9 and Corollary 4.4.6(a)]), $\Omega(M)/M$ has dimension at most $d-1$ since the multiplicity is the leading coefficient of the Hilbert polynomial. Then, the short exact sequence $0 \to M \to \Omega(M) \to \Omega(M)/M \to 0$ and the depth lemma [BH, Proposition 1.2.9] prove that $\Omega(M)/M$ is either zero or a Cohen–Macaulay module of dimension $d-1$.

It remains to compute the **ab**-index of $\Omega(M)/M$. For an **ab**-polynomial $\Psi = \sum_{S\subset[d]} a_S w_S \in \mathbb{Z}\langle\mathbf{a},\mathbf{b}\rangle$, we write $\pi(\Psi) = \sum_{S\subset[d]} a_S \mathbf{t}^S$. Since

$$\Psi_{\Omega(M)}(\mathbf{a},\mathbf{b}) - \Psi_M(\mathbf{a},\mathbf{b}) = \Psi_M(\mathbf{b},\mathbf{a}) - \Psi_M(\mathbf{a},\mathbf{b}) = \Upsilon\cdot(\mathbf{a}-\mathbf{b}),$$

we have

$$H_{\Omega(M)/M}(t_1,\dots,t_d) = \frac{\pi(\Upsilon\cdot(\mathbf{a}-\mathbf{b}))}{(1-t_1)\cdots(1-t_{d-1})(1-t_d)} = \frac{\pi(\Upsilon)}{(1-t_1)\cdots(1-t_{d-1})}$$

which shows that $\Psi_{\Omega(M)/M}(\mathbf{a},\mathbf{b}) = \Upsilon$. ☐

Let M be a squarefree P-module of dimension d. By considering the **b**-expression, we have the decomposition

$$\Psi_M(\mathbf{a},\mathbf{b}) = \Phi_{-1}\cdot\mathbf{c}^d + \Phi_0\cdot\mathbf{d}\mathbf{c}^{d-2} + \Phi_1\cdot\mathbf{d}\mathbf{c}^{d-3} + \cdots + \Phi_{d-2}\cdot\mathbf{d} + \Upsilon\cdot\mathbf{b}, \tag{11}$$

where $\Phi_{-1}\in\mathbb{Z}$, Φ_i is a **cd**-polynomial of degree i for $i = 0,1,\dots,d-2$ and Υ is the **cd**-polynomial of degree $d-1$ which appears in the **b**-expression. Then, for $k < d-1$, since the flag h-vector of the k-skeleton $M^{(k)}$ is given by $(h_S(M) : S \subset [k+1])$, we have

$$\Psi_{M^{(k)}}(\mathbf{a},\mathbf{b}) = \Phi_{-1}\cdot\mathbf{c}^{k+1} + \Phi_0\cdot\mathbf{d}\mathbf{c}^{k-1} + \cdots + \Phi_{k-1}\cdot\mathbf{d} + \Phi_k\cdot\mathbf{b}.$$

Observe that if M is Cohen–Macaulay then so is $M^{(k)}$ by Theorem 3.3. The above fact and Theorem 3.7 show

Corollary 5.4. *With the same notation as above, if M is a Cohen–Macaulay squarefree P-module over $\mathbb{R}$, then $\Psi_{\Omega(M^{(k)})/M^{(k)}}(\mathbf{a},\mathbf{b}) = \Phi_k$.*

By using Corollary 5.4, Karu [Ka] proved the non-negativity of the **cd**-indices of Gorenstein* posets. Moreover, Ehrenborg and Karu [EK, Theorem 5.6] proved that if M is a Cohen–Macaulay squarefree P-module and if the **ab**-index of M can be written in the form $\Psi_M(\mathbf{a},\mathbf{b}) = \Phi + \Upsilon\mathbf{a}$, then Φ is non-negative. (The statements are written in the language of sheaves.) Since **a**-expression always exists by Lemma 5.1, we have the following result.

Corollary 5.5. *Let M be a Cohen–Macaulay squarefree P-module over $\mathbb{R}$ of dimension d, and let*

$$\Psi_M(\mathbf{a},\mathbf{b}) = \Phi + \Upsilon\mathbf{a} = \Phi' + \Upsilon'\mathbf{b}$$

be the $\mathbf{a}$-expression and the $\mathbf{b}$-expression of $\Psi_M(\mathbf{a},\mathbf{b})$ respectively.

(i) *The coefficients of $\Phi(\mathbf{c},\mathbf{d})$ and $\Phi'(\mathbf{c},\mathbf{d})$ are non-negative.*
(ii) *If there is an injection $M \to \Omega(M)$ then the coefficients of $\Upsilon'(\mathbf{c},\mathbf{d})$ are non-negative.*
(iii) *If there is an injection $\Omega(M) \to M$ then the coefficients of $\Upsilon(\mathbf{c},\mathbf{d})$ are non-negative*

Proof. We first prove the non-negativity of Φ' by induction on d. The statement is obvious when $d = 0$. If $d = 1$, then the desired statement follows since

$$\Psi_M(\mathbf{a},\mathbf{b}) = h_\emptyset(M)\mathbf{a} + h_{\{1\}}(M)\mathbf{b} = h_\emptyset(M)\mathbf{c} + (h_{\{1\}}(M) - h_\emptyset(M))\mathbf{b}$$

and $h_\emptyset(M) = f_\emptyset(M) \geq 0$. For $d > 1$, the non-negativity of Φ' follows from Corollary 5.4 and the induction hypothesis.

The non-negativity of Φ follows from the duality of the $\mathbf{ab}$-index in Lemma 3.5 since it says $\Psi_{\Omega(M)}(\mathbf{a},\mathbf{b}) = \Psi_M(\mathbf{b},\mathbf{a}) = \Phi + \Upsilon\mathbf{b}$. Also, (ii) follows from (i) and Lemma 4.4, and, since $\Omega(\Omega(M)) \cong M$, (iii) follows from (ii) and the duality of the $\mathbf{ab}$-index. □

Remark 5.6. The coefficients of Φ' can be computed by embedding skeletons into its canonical module repeatedly. For example, if N is a Cohen–Macaulay squarefree P-module of dimension 8 with the $\mathbf{b}$-expression $\Psi_N = \Phi + \Upsilon\mathbf{b}$ and if γ is the coefficient of $\mathbf{c}^2\mathbf{dcdc}$ in Φ, then γ is obtained as follows: We note that, for any squarefree P-module M of dimension d, the coefficient of $\mathbf{c}^d$ in the $\mathbf{b}$-expression is equal to $\dim_K M_{\mathbf{0}}$. Let $N' = \Omega(N^{(5)})/N^{(5)}$. Then the coefficient of $\mathbf{c}^2\mathbf{dcdc}$ in Φ is the coefficient of $\mathbf{c}^2\mathbf{dc}$ in the $\mathbf{cd}$-index of N', and this is equal to the coefficient of $\mathbf{c}^2$ in the $\mathbf{cd}$-index of $N'' = \Omega(N'^{(2)})/N'^{(2)}$, which is equal to $\dim_K N''_{\mathbf{0}}$. This observation gives a ring-theoretic interpretation of [Ka, Theorem 4.10].

We give some examples of quasi CW-posets whose $\mathbf{a}$-expressions or $\mathbf{b}$-expressions are non-negative.

Example 5.7 (Ehrenborg–Karu)**.** If P is the face poset of a regular CW-complex Γ which is homeomorphic to a ball then $\Omega(K[P])$ is isomorphic to an ideal of $K[P]$ generated by all x_σ such that σ is an interior face (see [St3, II Theorem 7.3]). In this case, we have a natural injection $\Omega(K[P]) \to K[P]$. Thus, the $\mathbf{a}$-expression of $\Psi_P(\mathbf{a},\mathbf{b})$ is non-negative by Corollary 5.5(iii). Moreover, if $\Phi + \Upsilon\mathbf{a}$ is the $\mathbf{a}$-expression, then Υ is the $\mathbf{cd}$-index of the face poset of the boundary of Γ. See [EK, p. 231].

Example 5.8. A Cohen–Macaulay quasi CW-poset P is said to be *doubly Cohen–Macaulay* (over K) if, for any element $\sigma \in \widehat{P}$, the poset $P \setminus \{\sigma\}$ is Cohen–Macaulay (over K) and has the same rank as P. If P is doubly Cohen–Macaulay then there is an injection $K[P] \to \Omega(K[P])$ (see [St3, p. 91]). Thus doubly Cohen–Macaulay CW-posets have non-negative $\mathbf{b}$-expressions.

Now, we prove the main result of this section which implies Theorem 1.1 in the introduction.

Theorem 5.9. *Let M be a squarefree P-module of dimension d. There are unique* **cd**-*polynomials* $\Phi^{\mathbf{d}}, \Phi^{\mathbf{a}}, \Phi^{\mathbf{b}} \in \mathbb{Z}\langle \mathbf{c}, \mathbf{d} \rangle$ *such that*

$$\Psi_M(\mathbf{a}, \mathbf{b}) = \Phi^{\mathbf{d}} \cdot \mathbf{d} + \Phi^{\mathbf{a}} \cdot \mathbf{a} + \Phi^{\mathbf{b}} \cdot \mathbf{b}. \tag{12}$$

Moreover, if M is a Cohen–Macaulay squarefree P-module over $\mathbb{R}$ then all the coefficients in $\Phi^{\mathbf{d}}, \Phi^{\mathbf{a}}$ and $\Phi^{\mathbf{b}}$ are non-negative.

Proof. We first prove the uniqueness. If $\Psi_M(\mathbf{a}, \mathbf{b})$ can be written in the form $\Psi_M(\mathbf{a}, \mathbf{b}) = \Phi^{\mathbf{d}} \cdot \mathbf{d} + \Phi^{\mathbf{a}} \cdot \mathbf{a} + \Phi^{\mathbf{b}} \cdot \mathbf{b}$ then $(\Phi^{\mathbf{d}} \cdot \mathbf{d} + \Phi^{\mathbf{a}} \cdot \mathbf{c}) + (\Phi^{\mathbf{b}} - \Phi^{\mathbf{a}}) \cdot \mathbf{b}$ is the **b**-expression of $\Psi_M(\mathbf{a}, \mathbf{b})$. Then the uniqueness of $\Phi^{\mathbf{d}}, \Phi^{\mathbf{a}}$ and $\Phi^{\mathbf{b}}$ follows from the uniqueness of the **b**-expression.

Next, we prove the existence and non-negativity. Let $\Psi_M(\mathbf{a}, \mathbf{b}) = \Phi + \Upsilon \mathbf{b}$ be the **b**-expression of $\Psi_M(\mathbf{a}, \mathbf{b})$. If we write $\Phi = \Phi' \mathbf{c} + \Phi'' \mathbf{d}$, then

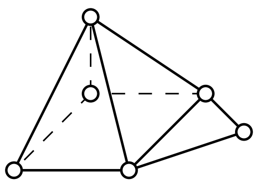
$$\Psi_M(\mathbf{a}, \mathbf{b}) = \Phi'' \cdot \mathbf{d} + \Phi' \cdot \mathbf{a} + (\Phi' + \Upsilon) \cdot \mathbf{b}.$$

This proves the existence of (12). Also, Corollary 5.5(i) implies that $\Phi^{\mathbf{a}} = \Phi'$ and $\Phi^{\mathbf{d}} = \Phi''$ are non-negative if M is Cohen–Macaulay. Finally, the non-negativity of $\Phi^{\mathbf{b}}$ follows from the duality $\Psi_M(\mathbf{a}, \mathbf{b}) = \Psi_{\Omega(M)}(\mathbf{b}, \mathbf{a})$. □

We call the right-hand side of (12) the *extended* **cd**-*index* of M (or of P if $M = K[P]$). Note that, if M has the symmetric flag h-vector, then $\Phi^{\mathbf{a}} = \Phi^{\mathbf{b}}$ and $\Phi^{\mathbf{a}}\mathbf{c} + \Phi^{\mathbf{d}}\mathbf{d}$ is the ordinary **cd**-index.

Remark 5.10. The existence of (12) holds for more general posets. Indeed, to prove Lemma 5.1, it suffices to assume that $\partial\sigma$ has the **cd**-index for each $\sigma \in \widehat{P}$. Thus, Lemma 5.1 and the existence of (12) hold for posets P such that $\langle \sigma \rangle$ is Eulerian [St4, Section 3.16] for all $\sigma \in \widehat{P}$.

Example 5.11. Let Δ be the 2-dimensional polyhedral complex obtained from the boundary of the square pyramid by gluing one triangle along an edge in the square (see the following Figure).



Let P be the face poset of Δ. Then P is Cohen–Macaulay, but is not Gorenstein*. Its flag f-vector and flag h-vector are given by

$$\sum_{S \subset [3]} f_S w_S = \mathbf{aaa} + 6\mathbf{baa} + 10\mathbf{aba} + 6\mathbf{aab} + 20\mathbf{bba} + 19\mathbf{bab} + 19\mathbf{abb} + 38\mathbf{bbb}$$

and

$$\Psi_P(\mathbf{a}, \mathbf{b}) = \mathbf{aaa} + 5\mathbf{baa} + 9\mathbf{aba} + 5\mathbf{aab} + 5\mathbf{bba} + 8\mathbf{bab} + 4\mathbf{abb} + \mathbf{bbb}.$$

The extended **cd**-index of P is

$$(4\mathbf{c})\mathbf{d} + (\mathbf{c}^2 + 4\mathbf{d})\mathbf{a} + (\mathbf{c}^2 + 3\mathbf{d})\mathbf{b}.$$

While the non-negativity of the extended **cd**-index of Cohen–Macaulay squarefree P-modules easily follows from Karu's results, it implies an interesting property on ordinary h-vectors. For a squarefree P-module M of dimension d, the *h-vector* of M is the vector $(h_0(M), h_1(M), \dots, h_d(M))$ defined by

$$h_i(M) = \sum_{S \subset [d], |S|=i} h_S(M).$$

By Lemma 2.5, this definition coincides with the usual definition of the h-vector. The next corollary proves Corollary 1.2.

Corollary 5.12. *Let M be a Cohen–Macaulay squarefree P-module of dimension d and $h(M) = (h_0, h_1, \dots, h_d)$. Then*

(i) $h_k \leq h_{d-1-k}$ *and* $h_{d-k} \leq h_{k+1}$ *for all* $0 \leq k < \frac{d}{2}$.

(ii) $h_{k-1} \leq h_k$ *for* $k \leq \frac{d}{2}$ *and* $h_k \geq h_{k+1}$ *for* $k \geq \frac{d}{2}$.

Proof. Let $\Psi_M(\mathbf{a}, \mathbf{b}) = \Phi^{\mathbf{d}} \cdot \mathbf{d} + \Phi^{\mathbf{a}} \cdot \mathbf{a} + \Phi^{\mathbf{b}} \cdot \mathbf{b}$ be the extended **cd**-index of M. Then, by substituting $\mathbf{a} = 1$, one obtains

$$\Psi_M(1, \mathbf{b}) = \Phi^{\mathbf{d}}(1 + \mathbf{b}, 2\mathbf{b}) \cdot 2\mathbf{b} + \Phi^{\mathbf{a}}(1 + \mathbf{b}, 2\mathbf{b}) + \Phi^{\mathbf{b}}(1 + \mathbf{b}, 2\mathbf{b}) \cdot \mathbf{b}. \tag{13}$$

On the other hand, by the definition of the h-vector, one has

$$\Psi_M(1, \mathbf{b}) = h_0(M) + h_1(M)\mathbf{b} + \cdots + h_d(M)\mathbf{b}^d. \tag{14}$$

For any homogeneous **cd**-polynomial $\Upsilon(\mathbf{c}, \mathbf{d}) \in \mathbb{Z}\langle \mathbf{c}, \mathbf{d} \rangle$ of degree k whose coefficients are non-negative, $\Upsilon(1 + \mathbf{b}, 2\mathbf{b})$ can be written in the form

$$\Upsilon(1 + \mathbf{b}, 2\mathbf{b}) = \alpha_0(1 + \mathbf{b})^k + \alpha_1\mathbf{b}(1 + \mathbf{b})^{k-2} + \alpha_2\mathbf{b}^2(1 + \mathbf{b})^{k-4} + \cdots$$

where $\alpha_0, \alpha_1, \alpha_2, \dots$ are non-negative integers. Thus if we write

$$\Upsilon(1 + \mathbf{b}, 2\mathbf{b}) = \gamma_0 + \gamma_1\mathbf{b} + \cdots + \gamma_k\mathbf{b}^k,$$

then $(\gamma_0, \gamma_1, \dots, \gamma_k)$ is a symmetric vector satisfying $\gamma_0 \leq \cdots \leq \gamma_{\frac{k}{2}} \geq \cdots \geq \gamma_k$ when k is even and $\gamma_0 \leq \cdots \leq \gamma_{\frac{k-1}{2}} = \gamma_{\frac{k+1}{2}} \geq \cdots \geq \gamma_k$ when k is odd. By applying these facts to (13) and (14) we obtain the desired property. $\square$

6. Lefschetz properties

Kubitzke and Nevo [KN, Theorem 1.1] proved that if Δ is the barycentric subdivision of a shellable simplicial complex of dimension $d - 1$, then there is an l.s.o.p. Θ of $K[\Delta]$ and a linear form w such that the multiplication map

$$\times w^{d-1-2k} : (K[\Delta]/(\Theta K[\Delta]))_k \to (K[\Delta]/(\Theta K[\Delta]))_{d-1-k}$$

is injective for $k \leq \frac{d-1}{2}$. By using this result, it was proved in [KN, Corollary 1.3] that if $(h_0, h_1, \dots, h_d)$ is the h-vector of the barycentric subdivision of a Cohen–Macaulay simplicial complex of dimension $d-1$, then $(h_0, h_1 - h_0, \dots, h_{\lfloor \frac{d}{2} \rfloor} - h_{\lfloor \frac{d}{2} \rfloor - 1})$ is an M-vector, that is, the Hilbert function of a graded K-algebra. The purpose of this section is to extend these results to barycentric subdivisions of Cohen–Macaulay polyhedral complexes.

We need the following fact which is a consequence of the Hard Lefschetz Theorem [St3, III Theorems 1.3 and 1.4].

Lemma 6.1. *If P is the face poset of a convex d-polytope, then there is an l.s.o.p. Θ of $\mathbb{R}[P]$ and a linear form $w \in \mathbb{R}[P]$ such that the multiplication map*

$$\times w^{d-2k} : (\mathbb{R}[P]/(\Theta\mathbb{R}[P]))_k \to (\mathbb{R}[P]/(\Theta\mathbb{R}[P]))_{d-k}$$

is bijective for $k \leq \frac{d}{2}$.

Proof. Let ρ be the maximal element of P corresponding to the convex polytope itself, and let $\partial P = P \setminus \{\rho\}$. Then $\mathbb{R}[P] = \mathbb{R}[\partial P][x_\rho]$ and $\mathbb{R}[\partial P]$ is the Stanley–Reisner ring of the barycentric subdivision of the boundary of a convex polytope. Since the barycentric subdivision of a convex polytope can be regarded as a convex polytope (see [ES]), it follows from [St3, III Theorem 1.3] that, by an appropriate choice of an l.s.o.p. Θ of $\mathbb{R}[\partial P]$, $\mathbb{R}[\partial P]/(\Theta\mathbb{R}[\partial P])$ is isomorphic to the cohomology ring of a toric variety arising from an integral simplicial d-polytope. Since x_ρ, Θ is an l.s.o.p. of $\mathbb{R}[P]$ and $\mathbb{R}[P]/((x_\rho,\Theta)\mathbb{R}[P]) \cong \mathbb{R}[\partial P]/(\Theta\mathbb{R}[\partial P])$, the desired statement follows from [St3, III Theorem 1.4]. □

We call a linear form w in Lemma 6.1 a *Lefschetz element* of $\mathbb{R}[P]/(\Theta\mathbb{R}[P])$.

Recall that a CW-poset P is said to be of *polyhedral type* if, for any $\sigma \in P$, $\langle\sigma\rangle$ is the face poset of a convex polytope. Obviously, the face poset of a polyhedral complex is a CW-poset of polyhedral type.

Theorem 6.2. *Let P be a CW-poset of polyhedral type, and let M be a Cohen–Macaulay squarefree P-module over $\mathbb{R}$ of dimension d. There is an l.s.o.p. Θ of M and a linear form w such that*

(i) *the multiplication*

$$\times w^{d-1-2k} : (M/\Theta M)_k \to (M/\Theta M)_{d-1-k}$$

is injective for $k \leq \frac{d-1}{2}$.

(ii) *the multiplication*

$$\times w^{d-1-2k} : (M/\Theta M)_{k+1} \to (M/\Theta M)_{d-k}$$

is surjective for $k \leq \frac{d-1}{2}$.

Proof. Let $R = \mathbb{R}[x_\sigma : \sigma \in \widehat{P}]$. We prove that, for a general choice of Θ and w, conditions (i) and (ii) hold. Since, for a $\mathbb{Z}$-graded R-module N and a linear form $w \in R$, the multiplication $\times w : N_k \to N_{k+1}$ is surjective if and only if $\times w : (N^T)_{-k-1} \to (N^T)_{-k}$ is injective, by Lemma 3.6 it suffices to prove (ii).

For a linear form $\theta = \sum_{\sigma\in\widehat{P}} \alpha_\sigma x_\sigma \in R$, let $\theta^{\leq k} = \sum_{\mathrm{rank}(\sigma)\leq k} \alpha_\sigma x_\sigma$. Take sufficiently general linear forms $\theta_1,\dots,\theta_{d+1} \in R$. Then, it follows from [Sw, Proposition 3.6] and Lemma 6.1 that, for each $\sigma \in \widehat{P}$ with $\mathrm{rank}\,\sigma = r$, $\theta_1^{\leq r},\dots,\theta_r^{\leq r}$ is an l.s.o.p. of $\mathbb{R}[\langle\sigma\rangle]$ and $\theta_{r+1}^{\leq r}$ is a Lefschetz element of $\mathbb{R}[\langle\sigma\rangle]/((\theta_1^{\leq r},\dots,\theta_r^{\leq r})\mathbb{R}[\langle\sigma\rangle])$.

Let $\Theta = \theta_1,\dots,\theta_d$. Consider the submodule $N = \bigoplus_{\sigma\in P_d} M_{\mathbf{e}_\sigma} R \subset M$. Since $M_{\mathbf{e}_\sigma} R \cong M_{\mathbf{e}_\sigma} \otimes_{\mathbb{R}} \mathbb{R}[\langle\sigma\rangle]$ for $\sigma \in P_d$ by Lemma 2.3, we have

$$N/(\Theta N) \cong \bigoplus_{\sigma\in P_d} \left(M_{\mathbf{e}_\sigma} \otimes_{\mathbb{R}} \left(\mathbb{R}[\langle\sigma\rangle]/(\Theta\mathbb{R}[\langle\sigma\rangle])\right)\right).$$

(Here elements of $M_{\mathbf{e}_\sigma}$ have degree $\mathbf{e}_\sigma$.) Thus the multiplication

$$\times \theta_{d+1}^{d-1-2k} : (N/\Theta N)_{k+1} \to (N/\Theta N)_{d-k}$$

is bijective for $k \leq \frac{d-1}{2}$. Consider the following commutative diagram

$$\begin{array}{ccc} (N/(\Theta N))_{k+1} & \longrightarrow & (M/(\Theta M))_{k+1} \\ \times\theta_{d+1}^{d-1-2k} \downarrow & & \downarrow \times\theta_{d+1}^{d-1-2k} \\ (N/(\Theta N))_{d-k} & \xrightarrow{\phi} & (M/(\Theta M))_{d-k}. \end{array}$$

To prove the surjectivity of the right vertical map, it is enough to prove that the lower horizontal map is surjective. Thus, we prove $\phi(N/(\Theta N)_k) = (M/(\Theta M))_k$ for $k \geq \frac{d+1}{2}$. For $\mathbf{u} = \sum_{\sigma \in \widehat{P}} u_\sigma \mathbf{e}_\sigma \in \mathbb{N}^{|\widehat{P}|}$, we write $\operatorname{rank}(\mathbf{u}) = \operatorname{rank}(\max(\operatorname{supp}(\mathbf{u})))$ and $|\mathbf{u}| = \sum_{\sigma \in \widehat{P}} u_\sigma$. Let $\mu \in M_{\mathbf{u}}$ with $\operatorname{rank}(\mathbf{u}) = r < d$ and with $|\mathbf{u}| \geq \frac{d+1}{2}$. We claim that

$$\mu \in \Theta M + \bigoplus_{\operatorname{rank}(\mathbf{v}) > r} M_{\mathbf{v}}.$$

Note that this claim implies the desired equation $\phi(N/(\Theta N)_k) = (M/(\Theta M))_k$ for $k \geq \frac{d+1}{2}$ since $\phi(N/(\Theta N)) = (\Theta M + \bigoplus_{\operatorname{rank} \mathbf{v} = d} M_{\mathbf{v}})/(\Theta M)$ by the definition of N.

Let $\sigma = \max(\operatorname{supp}(\mathbf{u}))$. By condition (b') of squarefree P-modules, $\mu = \tilde{\mu} x^{\mathbf{u}-\mathbf{e}_\sigma}$ for some $\tilde{\mu} \in M_{\mathbf{e}_\sigma}$ and

$$R/(\operatorname{ann} \tilde{\mu} + (x_\tau : \operatorname{rank} \tau > r)) \cong \mathbb{R}[\langle \sigma \rangle],$$

where $\operatorname{ann} \tilde{\mu} = \{f \in R : f\tilde{\mu} = 0\}$. Since $\theta_{r+1}^{\leq r}$ is a Lefschetz element of

$$\mathbb{R}[\langle \sigma \rangle]/\big((\theta_1^{\leq r}, \ldots, \theta_r^{\leq r})\mathbb{R}[\langle \sigma \rangle]\big) \cong R/(\operatorname{ann} \tilde{\mu} + (\theta_1^{\leq r}, \ldots, \theta_r^{\leq r}) + (x_\tau : \operatorname{rank} \tau > r)),$$

we have

$$(\operatorname{ann} \tilde{\mu} + (\theta_1^{\leq r}, \ldots, \theta_{r+1}^{\leq r}) + (x_\tau : \operatorname{rank} \tau > r))_k = R_k$$

for $k \geq \frac{r}{2}$. Since $r < d$, we have $\deg x^{\mathbf{u}-\mathbf{e}_\sigma} \geq \frac{d-1}{2} \geq \frac{r}{2}$. Thus we have

$$x^{\mathbf{u}-\mathbf{e}_\sigma} \in \operatorname{ann} \tilde{\mu} + (\theta_1^{\leq r}, \ldots, \theta_{r+1}^{\leq r}) + (x_\tau : \operatorname{rank} \tau > r).$$

Hence

$$\mu = \tilde{\mu} x^{\mathbf{u}-\mathbf{e}_\sigma} \in \big((\theta_1^{\leq r}, \ldots, \theta_{r+1}^{\leq r}) + (x_\tau : \operatorname{rank} \tau > r)\big)\,\tilde{\mu} \subset \Theta M + \bigoplus_{\operatorname{rank}(\mathbf{v}) > r} M_{\mathbf{v}},$$

as desired. □

A Cohen–Macaulay graded R-module M of dimension d is said to have the *weak Lefschetz property* (WLP for short) if there is an l.s.o.p. Θ of M and a linear form $w \in R$ such that the multiplication $\times w : (M/(\Theta M))_{k-1} \to (M/\Theta M))_k$ is either injective or surjective for all k. Theorem 6.2 implies the following corollary.

Corollary 6.3. *Let P be a CW-poset of polyhedral type and M a Cohen–Macaulay squarefree P-module of dimension d over $\mathbb{R}$. There is an l.s.o.p. Θ of M and a linear form w such that the multiplication $\times w : (M/(\Theta M))_{k-1} \to (M/\Theta M)_k$ is injective for $k \leq \frac{d}{2}$ and surjective for $k \geq \frac{d}{2} + 1$. In particular, M has the WLP if d is even.*

Remark 6.4. Theorem 6.2 and Corollary 6.3 hold over any infinite field if $\langle \sigma \rangle$ is the face poset of a simplex for any $\sigma \in \widehat{P}$ since Lemma 6.1 holds for simplices over any infinite field [KN, Proposition 2.3]. Thus, for barycentric subdivisions of simplicial complexes, one can work over positive characteristic. In particular, Corollary 6.3 solves the conjecture of Kubitzke and Nevo [KN, Conjecture 4.12] in odd dimensions.

Note that Corollary 6.3 cannot prove the WLP when d is odd since it says nothing about the multiplication map $\times w : (M/(\Theta M))_{\frac{d-1}{2}} \to (M/\Theta M))_{\frac{d+1}{2}}$.

Finally, we prove Theorem 1.3.

Proof of Theorem 1.3. Let P be a Cohen–Macaulay CW-poset of polyhedral type having rank n, and let $(h_0, h_1, \ldots, h_n)$ be the h-vector of Δ_P. By Corollary 6.3, there is an l.s.o.p. $\Theta = \theta_1, \ldots, \theta_n$ of $\mathbb{R}[P]$ and a linear form w such that

$$\begin{aligned}&\dim_{\mathbb{R}} \big(\mathbb{R}[P]/((w,\Theta)\mathbb{R}[P])\big)_k \\ &= \dim_{\mathbb{R}} \big(\mathbb{R}[P]/(\Theta\mathbb{R}[P])\big)_k - \dim_{\mathbb{R}} \big(\mathbb{R}[P]/(\Theta\mathbb{R}[P])\big)_{k-1} \\ &= h_k - h_{k-1}\end{aligned}$$

for $k \leq \frac{d}{2}$, where the second equality follows since $h_i = \dim_{\mathbb{R}}(\mathbb{R}[P]/(\Theta\mathbb{R}[P]))_i$ for all i (see e.g., [St3, II Corollary 2.5]). We prove that the Hilbert function of $\mathbb{R}[P]/((w,\Theta)\mathbb{R}[P])$ is the f-vector of a simplicial complex.

Let $\mathbb{R}[P] = \mathbb{R}[x_\sigma : \sigma \in \widehat{P}]/I$ and $c = |\widehat{P}| - n$. Since I is generated by monomials of degree 2, by the proof of [CCV, Theorem 2.1] the ideal I contains a regular sequence of the form $\ell_1\ell_2, \ldots, \ell_{2c-1}\ell_{2c}$, where $\ell_1, \ldots, \ell_{2c}$ are linear forms. Then, if we choose Θ sufficiently general, $\Lambda = \ell_1\ell_2, \ldots, \ell_{2c-1}\ell_{2c}, \theta_1, \ldots, \theta_n$ is also a regular sequence. Since $\mathbb{R}[P]/((w,\Theta)\mathbb{R}[P]) = \mathbb{R}[x_\sigma : \sigma \in \widehat{P}]/(I + (w,\Theta))$ and $I + (w,\Theta)$ contains Λ, the desired statement follows from Abedelfatah's result on the Eisenbud–Green–Harris conjecture [Ab, Corollary 4.3]. $\square$

7. Upper bounds for the **cd**-indices

In this section, we study upper bounds of the **cd**-indices of Gorenstein* posets. Billera and Ehrenborg [BE] proved that the **cd**-index of a d-polytope with v vertices are bounded above by the **cd**-index of a cyclic d-polytope with v vertices. Reading [Rea2, Section 7] study upper bounds of the **cd**-indices of Bruhat intervals in terms of the length of an interval. It is not possible to obtain upper bounds of the **cd**-indices of Gorenstein* posets for a fixed number of rank 1 elements or for a fixed rank since most coefficients of the **cd**-indices of Gorenstein* posets can be arbitrary large even if we fix their rank and the number of rank 1 elements. However, if we fix the number of rank i elements for all i, the **cd**-index is clearly bounded since its flag f-vector is bounded. The purpose of this section is to find sharp upper bounds of the **cd**-indices of Gorenstein* posets when we fix the number of rank i elements for all i.

We first study some algebraic properties of squarefree P-modules. We say that a squarefree P-module is *standard* if it is generated by elements of degree 0.

Lemma 7.1. *Let P be a quasi CW-poset and M a Cohen–Macaulay squarefree P-module of dimension d. Then, for any $k < d-1$, the module $\Omega(M^{(k)})$ is standard.*

Proof. Let

$$\mathcal{L}^M_\bullet : \ 0 \longrightarrow \mathcal{L}^M_d \xrightarrow{\partial_d} \mathcal{L}^M_{d-1} \xrightarrow{\partial_{d-1}} \mathcal{L}^M_{d-2} \xrightarrow{\partial_{d-2}} \cdots$$

be the Karu complex of M. Observe that each $\mathrm{Im}\,\partial_k$ is standard since $\mathcal{L}^M_k$ is the direct sum of Stanley–Reisner rings. Then the desired statement follows since $\Omega(M^{(k)}) \cong \ker \partial_{k+1} \cong \mathrm{Im}\,\partial_{k+2}$ by Theorem 3.1. $\square$

Remark 7.2. It is known that a proper skeleton of a Cohen–Macaulay CW-poset satisfying the intersection property is always doubly Cohen–Macaulay [Fl, Corollary 2.5]. (It is noteworthy that the definition of the doubly Cohen–Macaulay property in [Fl] is slightly different from the one given in Example 5.8, and "the intersection property" is really necessary in his context.) Lemma 7.1 gives an analogue of this fact for squarefree P-modules since a Cohen–Macaulay quasi CW-poset Q is doubly Cohen–Macaulay over K if and only if $\Omega(K[Q])$ is generated in degree 0 [St3, III Section 3].

For a squarefree P-module M, we write $\Phi_M^{\mathbf{d}} \cdot \mathbf{d} + \Phi_M^{\mathbf{a}} \cdot \mathbf{a} + \Phi_M^{\mathbf{b}} \cdot \mathbf{b}$ for its extended **cd**-index. Also, we write $\Phi_P^{\bullet} = \Phi_{K[P]}^{\bullet}$, where $\bullet$ is $\mathbf{a}, \mathbf{b}$ or $\mathbf{d}$.

Lemma 7.3. *Let P be a Cohen–Macaulay quasi CW-poset and M a Cohen–Macaulay standard squarefree P-module over $\mathbb{R}$ with $\dim M = \operatorname{rank} P$. Then we have the coefficientwise inequality*

$$\Phi_M^{\mathbf{d}} \cdot \mathbf{d} + \Phi_M^{\mathbf{a}} \cdot \mathbf{a} + \Phi_M^{\mathbf{b}} \cdot \mathbf{b} \leq (\dim_K M_{\mathbf{0}})(\Phi_P^{\mathbf{d}} \cdot \mathbf{d} + \Phi_P^{\mathbf{a}} \cdot \mathbf{a} + \Phi_P^{\mathbf{b}} \cdot \mathbf{b}).$$

Proof. Let $c = \dim_K M_{\mathbf{0}}$ and let $N = \bigoplus_{i=1}^{c} K[P]$ be the direct sum of c copies of $K[P]$. Since M is standard, there is a surjection $\pi : N \twoheadrightarrow M$. Then we have the following short exact sequence

$$0 \longrightarrow \ker \pi \longrightarrow N \xrightarrow{\pi} M \longrightarrow 0.$$

Observe that N and M are Cohen–Macaulay squarefree P-modules having the same dimension. It is clear that $\dim(\ker \pi) \leq \dim M$. It follows from the above short exact sequence and [BH, Proposition 1.2.9] that the depth of $\ker \pi$ is larger than or equal to the depth of M, which is equal to $\dim M$ since M is Cohen–Macaulay. Since the depth is smaller than or equal to the dimension [BH, Proposition 1.2.12], it follows that $\ker \pi$ is a Cohen–Macaulay squarefree P-module with $\dim(\ker \pi) = \dim M$. Consequently the short exact sequence says that the extended **cd**-index of N is the sum of those of $\ker \pi$ and M. Then the desired statement follows from the non-negativity of the extended **cd**-index. □

Lemma 7.3 has the following combinatorial meanings. It says that the extended **cd**-index of a Cohen–Macaulay regular CW-complex is larger than or equal to that of its Cohen–Macaulay subcomplex having the same dimension.

Corollary 7.4. *Let P be a Cohen–Macaulay quasi CW-poset and Q an order ideal of P having the same rank as P. If Q is Cohen–Macaulay, then we have the coefficientwise inequality*

$$\Phi_Q^{\mathbf{d}} \cdot \mathbf{d} + \Phi_Q^{\mathbf{a}} \cdot \mathbf{a} + \Phi_Q^{\mathbf{b}} \cdot \mathbf{b} \leq \Phi_P^{\mathbf{d}} \cdot \mathbf{d} + \Phi_P^{\mathbf{a}} \cdot \mathbf{a} + \Phi_P^{\mathbf{b}} \cdot \mathbf{b}.$$

Proof. Since $K[Q]$ is a standard squarefree P-module, the statement is the special case of Lemma 7.3 when $M = K[Q]$. □

For a homogeneous **cd**-polynomial $\Phi = \sum_{v \in \mathcal{B}_d} \alpha_v v \in \mathbb{Z}\langle \mathbf{c}, \mathbf{d} \rangle$ of degree d and for a **cd**-monomial u of degree $m < d$, let

$$\Phi_u = \sum_{v \in \mathcal{B}_{d-m}} \alpha_{vu} v.$$

Note that $\Phi_{\mathbf{dc}^k}$ is equal to Φ_{d-k-2} of (11) in Section 5.

Lemma 7.5. *Let P be a quasi CW-poset, M a Cohen–Macaulay squarefree P-module over $\mathbb{R}$ of dimension d and let $\Psi_M(\mathbf{a},\mathbf{b}) = \Phi + \Upsilon\mathbf{b}$ be the $\mathbf{b}$-expression of $\Psi_M(\mathbf{a},\mathbf{b})$. For any $\mathbf{cd}$-monomial of degree $\leq d$ of the form $u = \mathbf{d}u'$, there is a Cohen–Macaulay standard squarefree P-module N such that $\Psi_N(\mathbf{a},\mathbf{b}) = \Phi_u$.*

Proof. We may assume that u is of the form $u = \mathbf{d}\mathbf{c}^k$. Then Φ_u is the $\mathbf{cd}$-index of $N = \Omega(M^{(d-k-2)})/M^{(d-k-2)}$ by Corollary 5.4 and N is standard by Lemma 7.1. □

For a squarefree P-module M of dimension d with the $\mathbf{b}$-expression $\Psi_M(\mathbf{a},\mathbf{b}) = \Phi + \Upsilon\mathbf{b}$, we write

$$\alpha_S(M) = \alpha_S(\Phi)$$

for $S \in \mathcal{A}_d$, where $\alpha_S(\Phi)$ is as in the introduction. Also, we write $\alpha_S(P) = \alpha_S(K[P])$. Note that, for Gorenstein* posets, this definition coincides with that given in the introduction since the $\mathbf{b}$-expression of a Gorenstein* poset is equal to its $\mathbf{cd}$-index.

Lemma 7.6. *Let P be a Cohen–Macaulay quasi CW-poset and M a Cohen–Macaulay standard squarefree P-module over $\mathbb{R}$ of dimension d. Then, for any $S \in \mathcal{A}_d$, one has*

$$\alpha_S(M) \leq \alpha_\emptyset(M)\alpha_S(P).$$

Proof. Since M is a squarefree $P^{(d-1)}$-module and since $\alpha_S(P^{(d-1)}) = \alpha_S(P)$ for all $S \in \mathcal{A}_d$ by the computation given before Corollary 5.4, we may assume that $\dim M = \operatorname{rank} P$. Since $(\Phi_M^{\mathbf{a}} \cdot \mathbf{c} + \Phi_M^{\mathbf{d}} \cdot \mathbf{d}) + (\Phi_M^{\mathbf{b}} - \Phi_M^{\mathbf{a}}) \cdot \mathbf{b}$ is the $\mathbf{b}$-expression of $\Psi_M(\mathbf{a},\mathbf{b})$, the desired statement follows from Lemma 7.3. □

Now we prove the main result of this section.

Theorem 7.7. *Let P be a Cohen–Macaulay quasi CW-poset. If M is a Cohen–Macaulay standard squarefree P-module over $\mathbb{R}$ of dimension d, then $\alpha_S(M) \leq \alpha_\emptyset(M)\prod_{i\in S}\alpha_{\{i\}}(P)$ for all $S \in \mathcal{A}_d$.*

Proof. We prove the statement by induction on $|S|$. If $|S| = 0$, then the statement is obvious. Let $S \in \mathcal{A}_d$ with $|S| \geq 1$ and let $a = \min S$. By the induction hypothesis, $\alpha_{S\setminus\{a\}}(M) \leq \alpha_\emptyset(M)\prod_{i\in S\setminus\{a\}}\alpha_{\{i\}}(P)$. By Lemma 7.5, there is a Cohen–Macaulay standard squarefree P-module N such that $\alpha_\emptyset(N) = \alpha_{S\setminus\{a\}}(M)$ and $\alpha_{\{a\}}(N) = \alpha_S(M)$. Then Lemma 7.6 says

$$\alpha_S(M) = \alpha_{\{a\}}(N) \leq \alpha_\emptyset(N)\alpha_{\{a\}}(P) = \alpha_{S\setminus\{a\}}(M)\alpha_{\{a\}}(P) \leq \alpha_\emptyset(M)\prod_{i\in S}\alpha_{\{i\}}(P),$$

as desired. □

By considering the special case of Theorem 7.7 when $M = K[P]$, we obtain the following corollary which proves Theorem 1.4.

Corollary 7.8. *If P is a Cohen–Macaulay quasi CW-poset of rank n, then we have $\alpha_S(P) \leq \prod_{i\in S}\alpha_{\{i\}}(P)$ for all $S \in \mathcal{A}_n$.*

Recall that the *rank generating function* of a graded poset $P = \bigcup_{i=0}^n P_i$ is the polynomial $\sum_{k=0}^n f_{\{k\}}(P)t^k$, where $f_{\{0\}}(P) = 1$. For a Gorenstein* poset P of rank

n, Reading [Rea1, Theorem 2] proved that

$$\alpha_{\{k\}}(P) = -1 + \sum_{i=0}^{k} (-1)^{k-i} f_{\{i\}}(P)$$

for $k = 1, 2, \ldots, n-1$. These equations and the Euler relation $\sum_{i=0}^{n}(-1)^{n-i} f_{\{i\}}(P) = 1$ say that knowing the integers $\alpha_{\{1\}}(P), \ldots, \alpha_{\{n-1\}}(P)$ is equivalent to knowing $f_{\{1\}}(P), \ldots, f_{\{n\}}(P)$. Thus Corollary 7.8 gives upper bounds of the **cd**-indices of Gorenstein* posets for a fixed rank generating function. In the rest of this section, we prove that the bounds are sharp. To prove this we use the technique, which was called *unzipping* in [MN].

Let P be a Gorenstein* poset and $\sigma > \tau$ a cover relation with $\tau \neq \hat{0}$. We define the poset $\mathcal{U}(P; \sigma, \tau)$ as follows: delete the cover relation $\sigma > \tau$, add elements σ', τ' with $\mathrm{rank}(\sigma') = \mathrm{rank}(\sigma), \mathrm{rank}(\tau') = \mathrm{rank}(\tau)$ and add cover relations (i) $\sigma' < \rho$ for all cover relations $\sigma < \rho$, (ii) $\rho < \tau'$ for all cover relations $\rho < \tau$ and (iii) $\tau' < \sigma', \tau < \sigma'$ and $\tau' < \sigma$. The following result was shown in [Rea2, Theorem 4.6] and in [MN, Corollary 2.5].

Lemma 7.9. *Let P be a Gorenstein* poset of rank n and $\sigma > \tau$ a cover relation. Then $\mathcal{U}(P; \sigma, \tau)$ is Gorenstein* and*

$$\Phi_{\mathcal{U}(P;\sigma,\tau)}(\mathbf{c}, \mathbf{d}) = \Phi_P(\mathbf{c}, \mathbf{d}) + \Phi_{\partial\tau}(\mathbf{c}, \mathbf{d}) \cdot \mathbf{d} \cdot \Phi_{\mathrm{lk}_P(\sigma)}(\mathbf{c}, \mathbf{d}).$$

The next proposition guarantees that the bounds in Theorem 1.4 are sharp.

Proposition 7.10. *For any sequence $\alpha_1, \ldots, \alpha_{n-1}$ of nonnegative integers, there is a Gorenstein* poset P of rank n such that*

(i) $\alpha_S(P) = \prod_{i \in S} \alpha_i$ *for all* $S \in \mathcal{A}_n$.
(ii) *there is* $\sigma \in P_n$ *such that* $\alpha_S(\partial\sigma) = \prod_{i \in S} \alpha_i$ *for all* $S \in \mathcal{A}_{n-1}$.

Proof. We use induction on n. If $n = 1$, then there is nothing to prove. Also the statement is obvious when $n = 2$. Suppose $n > 2$. By the induction hypothesis, there is a Gorenstein* poset Q of rank $n-1$ and $\tau \in Q_{n-1}$ such that $\alpha_S(Q) = \prod_{i \in S} \alpha_i$ for all $S \in \mathcal{A}_{n-1}$ and $\alpha_S(\partial\tau) = \prod_{i \in S} \alpha_i$ for all $S \in \mathcal{A}_{n-2}$. Let $\Sigma Q = P \cup \{\eta, \eta'\}$ be the suspension of P. Thus ΣQ is the poset whose order is obtained from that of Q by adding the relations $\eta > \rho$ and $\eta' > \rho$ for all $\rho \in Q$. By [St2, Lemma 1.1], ΣQ is a Gorenstein* poset with $\Phi_{\Sigma Q}(\mathbf{c}, \mathbf{d}) = \Phi_Q(\mathbf{c}, \mathbf{d}) \cdot \mathbf{c}$.

If $\alpha_{n-1} = 0$ then the poset ΣQ satisfies the desired conditions (i) and (ii) since $\partial\eta' = Q$. Suppose $\alpha_{n-1} > 0$. Let $P(1) = \mathcal{U}(\Sigma Q; \eta, \tau)$ and let $\sigma(1) \in P(1)_n$ and $\tau(1) \in P(1)_{n-1}$ be the elements which are not in ΣQ. For $k = 2, 3, \ldots, \alpha_{n-1}$, we recursively define the poset $P(k) = \mathcal{U}(P(k-1); \sigma(k-1), \tau(k-1))$ and elements $\sigma(k) \in P(k)_n$ and $\tau(k) \in P(k)_{n-1}$ so that $\sigma(k)$ and $\tau(k)$ are the elements which are not in $P(k-1)$. We claim that $P = P(\alpha_{n-1})$ satisfies the desired conditions. By the construction of $P(k)$, $\partial\tau(k) = \partial\tau = \{\rho \in P : \rho < \tau\}$ in $P(k)$. Thus, by Lemma 7.9, we have

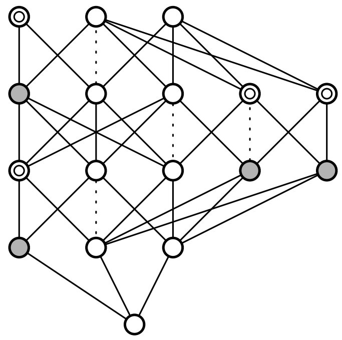
$$\Phi_P = \Phi_{\Sigma Q} + \alpha_{n-1}\Phi_{\partial\tau} \cdot \mathbf{d} = \Phi_Q \cdot \mathbf{c} + \alpha_{n-1}\Phi_{\partial\tau} \cdot \mathbf{d}.$$

Hence, for $S \in \mathcal{A}_n$,

$$\alpha_S(P) = \begin{cases} \alpha_S(Q), & \text{if } n-1 \notin S, \\ \alpha_{n-1} \cdot \alpha_{S \setminus \{n-1\}}(\partial\tau), & \text{if } n-1 \in S. \end{cases}$$

By the assumption on Q and τ, it follows that P satisfies condition (i). Also, since $\partial\eta' = Q$ in P, P also satisfies condition (ii). $\square$

Example 7.11. The Gorenstein* poset given in the proof of Proposition 7.10 is obtained from the face poset of the zero dimensional sphere (that is, the CW-complex consisting of two vertices) by taking suspensions and unzipping repeatedly. For example, if $\alpha_1 = \alpha_3 = 1$ and $\alpha_2 = 2$, then we obtain the following poset.



Dotted lines are the relations which are removed by unzipping. Double circles and colored elements are elements which are added by unzipping. Double circles correspond to $\sigma(-)$ and brown circles correspond to $\tau(-)$ in the proof.

8. Concluding remarks

On f-vectors. Recall that the f-vector $f(\Delta_P) = (f_{-1}, f_0, \dots, f_{n-1})$ of the order complex Δ_P of a poset P of rank n is given by $f_{i-1} = \sum_{S\subset[n],|S|=i} f_S(P)$. Considering Corollary 1.2, we ask the following question.

Problem 8.1. Is the f-vector of the order complex of a Cohen–Macaulay quasi CW-poset unimodal?

Brenti and Welker [BW] proved that, if P is a Cohen–Macaulay CW-poset of Boolean type, then the h-polynomial of Δ_P has only real zeros. This implies that its f-polynomial also has only real zeros, and therefore its f-vector is unimodal.

On flag f-vectors. Theorem 1.4 gives sharp upper bounds of flag f-vectors of Gorenstein* posets for a fixed rank generating function. However, we do not have an answer to the following problem.

Problem 8.2. Find sharp upper bounds of the flag f-vectors of (Cohen–Macaulay) CW-posets for a fixed rank generating function.

More strongly, the next problem would be of great interest.

Problem 8.3. Characterize all possible flag f-vectors of (Cohen–Macaulay) CW-posets.

The flag f-vectors of Gorenstein* posets of rank at most 4 were characterized in [MN]. Considering this fact, we think that Problem 8.3 will be tractable at least for CW-posets of rank 3.

On cd-indices. The proof of Theorem 7.7 says that if P is a Cohen–Macaulay quasi CW-poset of rank n, then, for a partition $S = \{a\} \cup T \in \mathcal{A}_n$ with $a = \min S$, one has $\alpha_S(P) \leq \alpha_{\{a\}}(P)\alpha_T(P)$. Moreover, the same argument proves $\alpha_S(P) \leq \alpha_{T_1}(P)\alpha_{T_2}(P)$ for any partition $S = T_1 \cup T_2$ with $\max T_1 < \min T_2$. We suggest the following conjecture which generalizes this property.

Conjecture 8.4. Let P be a Cohen–Macaulay CW-poset (or a Gorenstein* poset) of rank n and $S \in \mathcal{A}_n$. If $S = T_1 \cup T_2$ is a partition of S then $\alpha_S(P) \leq \alpha_{T_1}(P)\alpha_{T_2}(P)$.

Recently, Kalle Karu (personal communication) proved the above conjecture for S*-shellable CW-posets (see [St2] for the definition of S*-shellability). In particular, this shows that the conjecture holds for polytopes.

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