HESSENBERG VARIETIES AND ALGEBRAIC COMBINATORICS OF HYPERPLANE ARRANGEMENTS

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1. Introduction

Cohomologies of regular Hessenberg varieties are closely related to the algebra and combinatorics of hyperplane arrangements. This connection was first discovered by Sommers and Tymoczko [SoTy], and further developed by Abe et.al. [AHMMS]. The purpose of this chapter is to explain these connections between Hessenberg varieties and hyperplane arrangements, and in particular, to explain how algebra and combinatorics of hyperplane arrangements help understanding cohomologies of regular nilpotent Hessenberg varieties.

Before introducing general results, let us illustrate this connection briefly using simple examples. We will consider the Hessenberg functions

$$h = (3, 3, 3) \quad \text{and} \quad h' = (2, 3, 3),$$

and (type A_2) Hessenberg varieties associated with them. If we consider the definition of Hessenberg varieties in terms of Hessenberg spaces[1], we may consider that these Hessenberg varieties are defined from sets of roots

$$\{\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_2 - \mathbf{e}_3, \mathbf{e}_1 - \mathbf{e}_3\} \quad \text{and} \quad \{\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_2 - \mathbf{e}_3\}$$

in a type A_2 root system, where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are standard vectors in $\mathbb{R}^3$. Then we can naturally associate them with the hyperplane arrangements

$$\mathcal{A}_h = \{H_{12}, H_{23}, H_{13}\} \quad \text{and} \quad \mathcal{A}_{h'} = \{H_{12}, H_{23}\},$$

where $H_{ij} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_i = x_j\}$. The arrangements $\mathcal{A}_h$ and $\mathcal{A}_{h'}$ live in $\mathbb{R}^3$, but, since every hyperplane H_{ij} contains the linear line with the direction vector $(1, 1, 1)$, by restricting these arrangements to the plane $\{\boldsymbol{x} \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 6\}$ we can visualize them in $\mathbb{R}^2$ like Figure 1. From the figure, one can see that the arrangement $\mathcal{A}_h$ divide $\mathbb{R}^3$ into 6 areas, each of which is represented by the elements in $W = \{(i, j, k) \mid \{i, j, k\} = \{1, 2, 3\}\}$ (the set W may be identified with the symmetric group S_3 (type A_2 Weyl group)), and the arrangement $\mathcal{A}_{h'}$ divides $\mathbb{R}^3$ into 4 areas.

Now, consider the Poincaré polynomials of regular nilpotent Hessenberg varieties for Hessenberg functions h and h'. We know that their Poincaré polynomials are

[1]See §3.3 in Chapter 8 or §3.3 of this chapter for the definition of Hessenberg varieties. We note that Hessenberg variety associated with $h = (3, 3, 3)$ is the flag variety and Hessenberg variety associated with $h' = (2, 3, 3)$ is the Peterson variety.

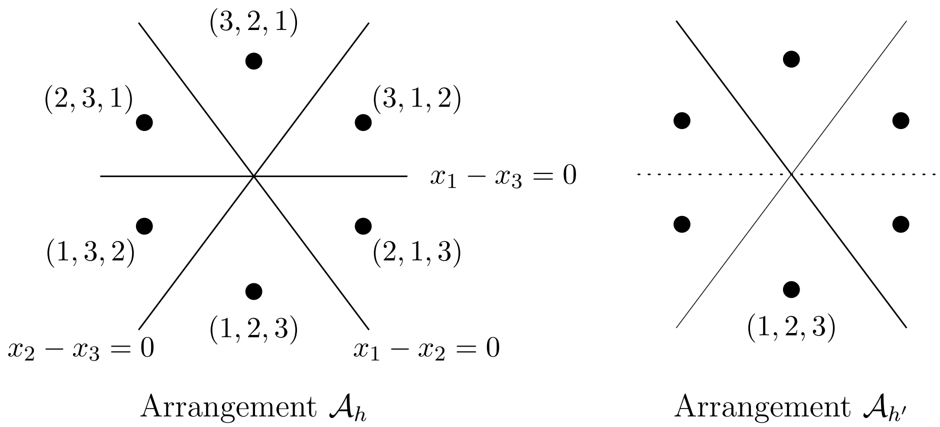


FIGURE 1. Visualization of $\mathcal{A}_h$ and $\mathcal{A}_{h'}$. The points are elements of W.

given by

$$\mathrm{Poin}\big(\mathrm{Hess}(\mathsf{N}, h), q\big) = 1 + 2q^2 + 2q^4 + q^6 \tag{1.1}$$

and

$$\mathrm{Poin}\big(\mathrm{Hess}(\mathsf{N}, h'), q\big) = 1 + 2q^2 + q^4, \tag{1.2}$$

where $\mathrm{Hess}(\mathsf{N}, -)$ denotes a regular nilpotent Hessenberg variety and $\mathrm{Poin}(-, q)$ denotes a Poincaré polynomial. Looking at Figure 1, one can see that these polynomials can be identified with "generating functions measuring distances of chambers" in the following sense: A **chamber** of a hyperplane arrangement $\mathcal{A}$ in $\mathbb{R}^n$ is a connected component of a complement of $\mathcal{A}$ in $\mathbb{R}^n$. Let's define the distance $\mathrm{dist}(R, R')$ between two chambers R and R' in a hyperplane arrangement $\mathcal{A}$ to be the number of hyperplanes in $\mathcal{A}$ that separate R and R'. Given a chamber R_0 of an arrangement $\mathcal{A}$, the polynomial

$$\mathrm{Dist}_{\mathcal{A}, R_0}(q) = \sum_{R\colon \text{chambers of } \mathcal{A}} q^{\mathrm{dist}(R, R_0)}$$

is called the **distance enumerator polynomial** of $\mathcal{A}$ with respect to the chamber R_0. Let F be the chamber of $\mathcal{A}_h$ (and of $\mathcal{A}_{h'}$) containing the point $(1, 2, 3)$. Then $\mathcal{A}_h$ has 1 chamber R with $\mathrm{dist}(R, F) = 0$, 2 chambers R with $\mathrm{dist}(R, F) = 1$, 2 chambers R with $\mathrm{dist}(R, F) = 2$, and 1 chamber R with $\mathrm{dist}(R, F) = 3$, so we have

$$\mathrm{Dist}_{\mathcal{A}_h, F}(q) = 1 + 2q + 2q^2 + q^3. \tag{1.3}$$

Similarly, the arrangement $\mathcal{A}_{h'}$ has 1 chamber R with $\mathrm{dist}(R, F) = 0$, 2 chambers R with $\mathrm{dist}(R, F) = 1$, and 1 chamber R with $\mathrm{dist}(R, F) = 2$, so

$$\mathrm{Dist}_{\mathcal{A}_{h'}, F}(q) = 1 + 2q + q^2. \tag{1.4}$$

Comparing (1.1)–(1.4), you will notice that, by replacing q^2 with q, the Poincaré polynomials coincide with these distance enumerator polynomials!

We can also see a similar phenomenon for a regular semisimple Hessenberg variety $\mathrm{Hess}(\mathsf{S},-)$. We know that

$$\mathrm{Poin}\big(\mathrm{Hess}(\mathsf{S},h),q\big)=\mathrm{Poin}\big(\mathrm{Hess}(\mathsf{N},h),q\big)=1+2q^2+2q^4+q^6 \tag{1.5}$$

and

$$\mathrm{Poin}\big(\mathrm{Hess}(\mathsf{S},h'),q\big)=1+4q^2+q^4. \tag{1.6}$$

These polynomials can be identified with the distance enumerator polynomials of the set W of points drawn in Figure 1. Indeed, for the arrangement $\mathcal{A}_h$, 1 point of W lies in the chamber of distance 0, 2 points of W lie in chambers of distance 1, 2 points of W lie in chambers of distance 2, and 1 point lies in the chamber of distance 3. Similarly, for the arrangement $\mathcal{A}_{h'}$, 1 point of W lies in the chamber of distance 0, 4 points of W lie in chambers of distance 1, and 1 point of W lies in the chamber of distance 2. So we get polynomials $1+2q+2q^2+q^3$ and $1+4q+q^2$ (compare these with (1.5) and (1.6)) as generating functions.

This agreement of Poincaré polynomials and distance enumerator polynomials is not a coincidence. Indeed, we will see in section 3 that this happens for all regular nilpotent and semisimple Hessenberg varieties of any Lie type.

For regular nilpotent Hessenberg varieties, we can actually say more. The ring structure of the cohomology of a regular nilpotent Hessenberg variety can be determined from "vector fields of hyperplane arrangements". Let $\mathcal{D}_h$ and $\mathcal{D}_{h'}$ be the polynomial vector fields tangent to hyperplanes in $\mathcal{A}_h$ and in $\mathcal{A}_{h'}$ respectively. So, by setting $\mathcal{R}=\mathbb{R}[x_1,x_2,x_3]$,

$$\mathcal{D}_h=\left\{\big(f(\boldsymbol{x}),g(\boldsymbol{x}),h(\boldsymbol{x})\big)\in\mathcal{R}^3 \;\middle|\; \begin{array}{l}\big(f(\boldsymbol{a}),g(\boldsymbol{a}),h(\boldsymbol{a})\big)\in H_{12} \text{ if } \boldsymbol{a}\in H_{12}\\ \big(f(\boldsymbol{a}),g(\boldsymbol{a}),h(\boldsymbol{a})\big)\in H_{23} \text{ if } \boldsymbol{a}\in H_{23}\\ \big(f(\boldsymbol{a}),g(\boldsymbol{a}),h(\boldsymbol{a})\big)\in H_{13} \text{ if } \boldsymbol{a}\in H_{13}\end{array}\right\}$$

and

$$\mathcal{D}_{h'}=\left\{\big(f(\boldsymbol{x}),g(\boldsymbol{x}),h(\boldsymbol{x})\big)\in\mathcal{R}^3 \;\middle|\; \begin{array}{l}\big(f(\boldsymbol{a}),g(\boldsymbol{a}),h(\boldsymbol{a})\big)\in H_{12} \text{ if } \boldsymbol{a}\in H_{12}\\ \big(f(\boldsymbol{a}),g(\boldsymbol{a}),h(\boldsymbol{a})\big)\in H_{23} \text{ if } \boldsymbol{a}\in H_{23}\end{array}\right\},$$

where $f(\boldsymbol{x})=f(x_1,x_2,x_3)$. One can easily check that $\mathcal{D}_h$ and $\mathcal{D}_{h'}$ are $\mathcal{R}$-submodules of $\mathcal{R}^3$, and by a careful computation one can see that $\mathcal{D}_h$ is generated by the following three elements

$$\big(x_1^2,x_2^2,x_3^2\big),\ \big(x_1,x_2,x_3\big),\ (1,1,1), \tag{1.7}$$

and similarly $\mathcal{D}_{h'}$ is generated by

$$\big(x_1-x_2,0,0\big),\ \big(x_2-x_3,x_2-x_3,0\big),\ (1,1,1). \tag{1.8}$$

On the other hand, we know (since $\mathrm{Hess}(\mathsf{N},h)$ is a flag variety) that

$$H^*\big(\mathrm{Hess}(\mathsf{N},h)\big)\cong\mathcal{R}/\big(x_1^3+x_2^3+x_3^3,x_1^2+x_2^2+x_3^2,x_1+x_2+x_3\big)$$

(cohomologies are with real coefficients). Also, using a recursive presentation of cohomology rings of type A regular nilpotent Hessenberg varieties given in Chapter 9, one can see

$$H^*\big(\mathrm{Hess}(\mathsf{N},h')\big)\cong\mathcal{R}/\big((x_1-x_2)x_1,(x_2-x_3)(x_1+x_2),x_1+x_2+x_3\big).$$

From these formulas, we can see the following surprising relation between vector fields of hyperplane arrangements and cohomology rings of regular nilpotent Hessenberg varieties: Each of $H^*(\mathrm{Hess}(\mathsf{N},h))$ and $H^*(\mathrm{Hess}(\mathsf{N},h'))$ has three generating relations, and these three relations are nothing but the inner products of generators of vector fields with (x_1,x_2,x_3)! Indeed, the three generating relations of $H^*(\mathrm{Hess}(\mathsf{N},h))$ can be written in the form

$$\begin{array}{ll} x_1^3+x_2^3+x_3^3 & =\big(x_1^2,x_2^2,x_3^2\big)\cdot(x_1,x_2,x_3) \\ x_1^2+x_2^2+x_3^2 & =\big(x_1,x_2,x_3\big)\cdot(x_1,x_2,x_3) \\ x_1+x_2+x_3 & =(1,1,1)\cdot(x_1,x_2,x_3). \end{array}$$

Also, the three generating relations of $H^*(\mathrm{Hess}(\mathsf{N},h'))$ can be written in the form

$$\begin{array}{ll} (x_1-x_2)x_1 & =\big((x_1-x_2),0,0\big)\cdot(x_1,x_2,x_3) \\ (x_2-x_3)(x_1+x_2) & =\big((x_2-x_3),(x_2-x_3),0\big)\cdot(x_1,x_2,x_3) \\ x_1+x_2+x_3 & =(1,1,1)\cdot(x_1,x_2,x_3). \end{array}$$

Again, this is not a coincidence. We will see in Section 7 that this happens for all regular nilpotent Hessenberg varieties.

The above nice relations between hyperplane arrangements and Hessenberg varieties provide several outcomes. For example, from (1.1) and (1.2), one can see the factorizations of Poincaré polynomials

$$\mathrm{Poin}\big(\mathrm{Hess}\big(\mathsf{N},(3,3,3)\big),q\big)=1\times(1+q^2)\times(1+q^2+q^4)$$

and

$$\mathrm{Poin}\big(\mathrm{Hess}\big(\mathsf{N},(2,3,3)\big),q\big)=1\times(1+q^2)\times(1+q^2).$$

We will see in Section 7 that this kind of factorization holds for any regular nilpotent Hessenberg variety using alegbraic and combinatorial results in hyperplane arrangement theory. We will also see in Section 8 that vector fields of hyperplane arrangements enable us to find a concrete presentation of the cohomology ring of any regular nilpotent Hessenberg variety as the quotient of a polynomial ring.

What is the advantage of considering hyperplane arrangements? One of the most important features is that hyperplane arrangements provide a nice combinatorial way to study cohomologies of Hessenberg varieties, and in particular, provide us with geometric intuition to study them. We expect that a hyperplane arrangement theoretic viewpoint will bring more connections among the algebra, combinatorics and geometry of Hessenberg varieties to light in future work.

Organization of this article: In Section 2, we review some basic results on combinatorics of hyperplane arrangements such as intersection posets, characteristic polynomials, Zaslabsky's theorem and posets of chambers. Then in Section 3 we discuss chambers of subarrangements of Weyl arrangements. At the end of this section, we will see that Poincaré polynomials of regular nilpotent Hessenberg varieties coincide with distance enumerator polynomials of the corresponding hyperplane arrangements. Sections 4,5,6 and 7 are devoted to explaining the connection between vector fields of hyperplane arrangements and cohomology rings of regular nilpotent Hessenberg varieties. In the first two sections, we discuss some algebraic properties of vector fields of arrangements. In Section 4, we review the theory of free arrangements and explain how the freeness of Weyl arrangements relates to a classical invariant theory. In Section 5, we introduce the Solomon–Terao algebra which can be considered as an arrangement theoretic generalization of coinvariant algebras. Then after reviewing some results on torus equivariant cohomologies of flag varieties and regular nilpotent Hessenberg varieties in Section 6, we show in Section 7 that cohomology rings of regular nilpotent Hessenberg varieties can be computed from vector fields of corresponding hyperplane arrangements. In Section 8, we give explicit presentations of cohomology rings of regular nilpotent Hessenberg varieties for classical types using vector fields. We also collect some basic results on regular sequences in the Appendix for people who are not familiar with commutative algebra. The first two sections mainly discuss combinatorial aspects of hyperplane arrangements, while the latter five sections will focus on algebraic aspects.

Note. A connection between hyperplane arrangements and Hessenberg varieties was first investigated by Sommers and Tymoczko [SoTy]. They defined "ideal exponents" for a lower ideal in a root system, generalizing the usual exponents of a root system, and posed two conjectures indicating that ideal exponents are related to both Poincaré polynomials of regular nilpotent Hessenberg varieties and freeness of hyperplane arrangements associated with lower ideals in a root system. Motivated by this work, properties of hyperplane arrangements associated with lower ideals have been further investigated, and in particular the conjecture on freeness of arrangements was solved in [ABCHT]. On the other hand, a connection between vector fields and cohomologies of regular nilpotent Hessenberg varieties was found by Abe et.al. [AHMMS] inspired from studies on rings structures of cohomologies of Hessenberg varieties (see Chapter 9 and references therein for backgrounds and more results on this topic), and this connection solves the other conjecture of Sommers and Tymoczko on Poincaré polynomials. It was also used to obtain an explicit presentation of the cohomology ring of a regular nilpoent Hssenberg variety of any Lie type as a quotient of a polynomial ring [EHNT].

Notation list

We list notation which we often use in this chapter for later convenience. All cohomologies will be taken with coefficients over $\mathbb{R}$ throughout this chapter.

- $[n] = \{1, 2, \ldots, n\}$: set of integers $1, 2, \ldots, n$
- V: n-dimensional Euclidian space
- $\mathbf{e}_1, \ldots, \mathbf{e}_n$: orthonormal basis for V
- $x_1, \ldots, x_n$: dual basis of $\mathbf{e}_1, \ldots, \mathbf{e}_n$
- $\mathcal{R} = \mathrm{sym}(V^*) = \mathbb{R}[x_1, \ldots, x_n]$: symmetric algebra of V^*
- $\mathrm{Hilb}(M, q) = \sum_{k\geq 0}(\dim M_k)q^k$: Hilbert series of a graded $\mathcal{R}$-module M
- $\mathcal{A}$: (linear) hyperplane arrangement in V
- H_α: hyperplane orthogonal to $\alpha \in V \setminus \{\mathbf{0}\}$
- $\ell_\alpha = a_1x_1 + \cdots + a_nx_n$: linear form associated with $\alpha = a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n$
- s_α: reflection defined by the hyperplane H_α.
- G: simple linear algebraic group
- Φ: root system

- W: Weyl group
- $H^*(X)$: cohomology of X with coefficients in $\mathbb{R}$
- $\mathrm{Poin}(X,q)=\sum_{k\geq 0}(\dim_{\mathbb{R}} H^k(X))q^k$: Poincaré polynomial of X
- $\mathrm{Hess}(\mathsf{N},I)$: regular nilpotent Hessenberg variety for a lower ideal I
- $\mathrm{Hess}(\mathsf{S},I)$: regular semisimple Hessenberg variety for a lower ideal I

2. A quick intro to combinatorics of hyperplane arrangements

In this section we quickly review basic combinatorial results and techniques in hyperplane arrangement theory.

2.1. **Definition and notation.** Let $V \cong \mathbb{R}^n$ be an n-dimensional Euclidian space with an inner product $(-,-)$. We fix an orthonormal basis $\mathbf{e}_1,\dots,\mathbf{e}_n$ for V and its dual basis $x_1,\dots,x_n$ in the dual space V^*. We write $\mathcal{R}=\mathrm{sym}(V^*)=\mathbb{R}[x_1,\dots,x_n]$ for the symmetric algebra of V^*. A **hyperplane** in V is a codimension 1 affine subspace of V. Any hyperplane H in V is a kernel of a non-zero linear form $a_1x_1+\cdots+a_nx_n-b\in\mathcal{R}$, where $a_1,\dots,a_n,b\in\mathbb{R}$. More informally we may consider that it is a subspace of V defined by the equation

$$a_1x_1+\cdots+a_nx_n-b=0.$$

This linear form $a_1x_1+\cdots+a_nx_n-b$ is called a **defining linear form** of H. A defining linear form of H is unique up to a non-zero scalar multiple, and we often use ℓ_H to denote a (fixed) defining linear form of H. A hyperplane arrangement $\mathcal{A}$ in V is a finite collection of hyperplanes in V. A hyperplane arrangement $\mathcal{A}$ is said to be **linear** if every hyperplane in $\mathcal{A}$ is a linear space (i.e. contains the origin). While many of the results in this section hold for an arbitrary hyperplane arrangement, we only consider linear hyperplane arrangements to simplify proofs. We refer the readers to [St07, OT] for results in a general setting. In the rest of this chapter, when we write "**arrangement**" it always means a linear hyperplane arrangement.

Hyperplane arrangements are often described by their defining polynomials. For an arrangement $\mathcal{A}$, the polynomial

$$Q(\mathcal{A}):=\prod_{H\in\mathcal{A}}\ell_H,$$

where ℓ_H is a defining linear form of H, is called a **defining polynomial** of $\mathcal{A}$. For example, consider the arrangement $\mathcal{A}=\{H_1,H_2,H_3,H_4\}$ in $\mathbb{R}^2$ defined by

$$H_1: x_1=0,\quad H_2: x_2=0,\quad H_3: x_1-x_2=0,\quad H_4: x_1+x_2=0.$$

Then $x_1x_2(x_1-x_2)(x_1+x_2)$ is a defining polynomial of $\mathcal{A}$. See Figure 2.

2.2. **Characteristic polynomials.** Next, we introduce the characteristic polynomials of an arrangement, which is one of the most important combinatorial invariant in the hyperplane arrangement theory.

We first define intersection posets and the Möbius function on it. Let $\mathcal{A}$ be an arrangement in V. The **intersection poset** $\mathcal{L}_{\mathcal{A}}$ (also called the **intersection lattice**) is the poset whose elements are the subspaces of V obtained as an intersection

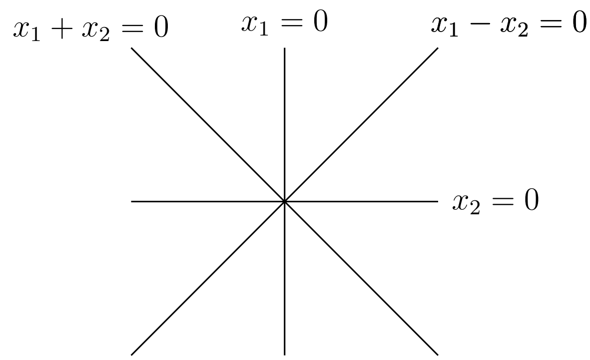


FIGURE 2. Arrangement with a defining polynomial $x_1x_2(x_1 - x_3)(x_1 + x_3)$

of hyperplanes in $\mathcal{A}$ and whose order is defined by the reverse inclusion. Thus, as a set,

$$\mathcal{L}_\mathcal{A} = \{H_1 \cap \cdots \cap H_k \mid \{H_1, \ldots, H_k\} \subset \mathcal{A}\},$$

where we consider that V itself is an element of $\mathcal{L}_\mathcal{A}$. Clearly V is the unique smallest element of $\mathcal{L}_\mathcal{A}$ and we write $\hat{0} = V$ when we consider it as an element of $\mathcal{L}_\mathcal{A}$. Let

$$I(\mathcal{L}_\mathcal{A}) = \{(a, b) \in \mathcal{L}_\mathcal{A} \times \mathcal{L}_\mathcal{A} \mid a \leq b\}.$$

The **Möbius function** on $\mathcal{L}_\mathcal{A}$ is the function $\mu : I(\mathcal{L}_\mathcal{A}) \to \mathbb{Z}$ defined inductively as follows[2]

- $\mu(a, a) = 1$ for any $a \in \mathcal{L}_\mathcal{A}$,
- $\sum_{a \leq c \leq b} \mu(a, c) = 0$ for all $(a, b) \in I(\mathcal{L}_\mathcal{A})$ with $a \neq b$.

We actually only consider values $\mu(\hat{0}, y)$ with $y \in \mathcal{L}_\mathcal{A}$, and such a value can be computed as follows:

- $\mu(\hat{0}, \hat{0}) = 1$,
- $\mu(\hat{0}, H) = -\mu(\hat{0}, \hat{0}) = -1$ for any $H \in \mathcal{A}$,
- $\mu(\hat{0}, y) = -\sum_{H \in \mathcal{A},\ H \supset y} \mu(\hat{0}, H) - \mu(\hat{0}, \hat{0})$ for any codim 2 subspace $y \in \mathcal{L}_\mathcal{A}$,

$$\vdots$$

Definition 2.1. Let $\mathcal{A}$ be an arrangement in V. The polynomial in q

$$\chi_\mathcal{A}(q) := \sum_{y \in \mathcal{L}_\mathcal{A}} \mu(\hat{0}, y) q^{\dim y}$$

is called the **characteristic polynomial** of $\mathcal{A}$.

Example 2.2. Consider the arrangement $\mathcal{A} = \{H_1, H_2, H_3, H_4\}$ in Figure 2. Then $\mathcal{L}_\mathcal{A} = \{\{\mathbf{0}\}, H_1, H_2, H_3, H_4, \hat{0} = V\}$. Figure 3 presents the Hasse diagram of the intersection poset of $\mathcal{A}$ and values of $\mu(\hat{0}, y)$ for each $y \in \mathcal{L}_\mathcal{A}$. The characteristic polynomial of A is given by

$$\chi_\mathcal{A}(q) = 1 \cdot q^2 + (-1) \cdot q + (-1) \cdot q + (-1) \cdot q + (-1) \cdot q + 3 \cdot q^0 = q^2 - 4q + 3.$$

[2]Möbius functions can be defined for more general posets. See [St12, §3.7] for a general theory.

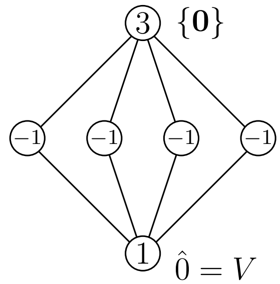


FIGURE 3. Intersection poset of the arrangement in Figure 2.

2.3. **Deletion and restriction.** We next introduce the deletion and restriction formula, which is a basic but powerful tool to study characteristic polynomials. Let $\mathcal{A}$ be an arrangement in V and $H \in \mathcal{A}$. The **deletion** $\mathcal{A} \setminus H$ is the arrangement

$$\mathcal{A} \setminus H = \mathcal{A} \setminus \{H\}.$$

The **restriction** A/H is the arrangement in $H \cong \mathbb{R}^{n-1}$ defined by[3]

$$\mathcal{A}/H = \{H \cap H' \mid H' \in \mathcal{A} \setminus H\}.$$

Proposition 2.3 (Deletion-restriction formula)**.** *If $\mathcal{A}$ is an arrangement in V and $H_0 \in \mathcal{A}$, then*

$$\chi_{\mathcal{A}}(q) = \chi_{\mathcal{A}\setminus H_0}(q) - \chi_{\mathcal{A}/H_0}(q).$$

Proof. We first prove the following formula due to Whitney

$$\chi_{\mathcal{A}}(q) = \textstyle\sum_{\mathcal{B}\subset\mathcal{A}} (-1)^{|\mathcal{B}|}\, q^{\dim(\bigcap_{H\in\mathcal{B}} H)}, \tag{2.1}$$

where $|X|$ denotes the cardinality of a finite set X and $\mathcal{B}$ could be an empty set in the summation. For any $y \in \mathcal{L}_{\mathcal{A}}$, by setting $\mathcal{A}' = \{H \in \mathcal{A} \mid H \supset y\}$, we have

$$\sum_{\hat{0}\le z\le y} \left(\sum_{\mathcal{B}\subset\mathcal{A},\ (\bigcap_{H\in\mathcal{B}} H)=z} (-1)^{|\mathcal{B}|} \right) = \sum_{\mathcal{B}\subset\mathcal{A}'} (-1)^{|\mathcal{B}|} = \begin{cases} 1 & \text{if } y = \hat{0}, \\ 0 & \text{if } y \neq \hat{0}. \end{cases}$$

Let $\phi(z) = \sum_{\mathcal{B}\subset\mathcal{A},\ (\bigcap_{H\in\mathcal{B}} H)=z} (-1)^{|\mathcal{B}|}$. By the above equation, we have $\phi(\hat{0}) = 1$ and $\sum_{\hat{0}\le z\le y} \phi(z) = 0$ for $y \neq \hat{0}$. This tells that $\phi(z)$ coincides with $\mu(\hat{0}, z)$ by the definition of the Möbius function, so we have

$$\sum_{\mathcal{B}\subset\mathcal{A},\ (\bigcap_{H\in\mathcal{B}} H)=z} (-1)^{|\mathcal{B}|} = \mu(\hat{0}, z)$$

for any $z \in \mathcal{L}_{\mathcal{A}}$. This clearly proves (2.1).

Now by (2.1) we have

$$\begin{aligned} \chi_{\mathcal{A}}(q) &= \sum_{\mathcal{B}\subset\mathcal{A},\ H_0\notin\mathcal{B}} (-1)^{|\mathcal{B}|} q^{\dim(\bigcap_{H\in\mathcal{B}} H)} + \sum_{\mathcal{B}\subset\mathcal{A},\ H_0\in\mathcal{B}} (-1)^{|\mathcal{B}|} q^{\dim(\bigcap_{H\in\mathcal{B}} (H\cap H_0))} \\ &= \chi_{\mathcal{A}\setminus H_0}(q) - \chi_{\mathcal{A}/H_0}(q), \end{aligned}$$

as desired. □

[3]Since we assume that $\mathcal{A}$ is linear, $H \cap H'$ is indeed a hyperplane of H for any $H' \in \mathcal{A} \setminus H$.

2.4. **Zaslavsky's theorem.** Let $\mathcal{A}$ be an arrangement in $V \cong \mathbb{R}^n$ and let $M(\mathcal{A}) = V \setminus \bigcup_{H\in\mathcal{A}} H$ be the complement of the hyperplanes in $\mathcal{A}$. A connected component of $M(\mathcal{A})$ is called a **chamber** of $\mathcal{A}$. We write $C(\mathcal{A})$ for the set of all chambers of $\mathcal{A}$. Zaslabsky's theorem below gives an interesting connection between characteristic polynomials and the numbers of chambers of arrangements.

Theorem 2.4 (Zaslavsky's theorem)**.** *If $\mathcal{A}$ is an arrangement in V, then*

$$|C(\mathcal{A})| = (-1)^n \chi_{\mathcal{A}}(-1).$$

Proof. The statement is clear when $\mathcal{A} = \varnothing$ since $\mathcal{L}_\varnothing = \{V\}$ and $\chi_\varnothing(q) = q^n$. Assume $\mathcal{A} \neq \varnothing$ and take $H \in \mathcal{A}$. Clearly, the number of chambers of $\mathcal{A}$ equals to "the number of chambers of $\mathcal{A} \setminus H$" plus "the number of chambers of $\mathcal{A} \setminus H$ that intersect H". The latter equals to the number of chambers of $\mathcal{A}/H$, so we have the deletion-restriction formula

$$|C(\mathcal{A})| = |C(\mathcal{A} \setminus H)| + |C(\mathcal{A}/H)|$$

for the numbers of chambers. Then the desired assertion follows from the deletion-restriction formula for characteristic polynomials using induction on the cardinality of $\mathcal{A}$. □

Example 2.5. Consider the arrangement $\mathcal{A} = \{H_1, H_2, H_3, H_4\}$ in Figure 2. We already see that $\chi_{\mathcal{A}}(t) = t^2 - 4t + 3$. Figure 2 clearly shows that the number of chambers of $\mathcal{A}$ is 8, which is equal to $(-1)^2\chi_{\mathcal{A}}(-1) = 1 + 4 + 3$.

Remark 2.6. If $\mathcal{A}$ is an arrangement in $V \cong \mathbb{R}^n$, then it is known that $\chi_{\mathcal{A}}(t)$ must be a polynomial of the form

$$\chi_{\mathcal{A}}(t) = c_n t^n - c_{n-1}t^{n-1} + c_{n-2}t^{n-2} - \cdots$$

with $c_0, c_1, \ldots, c_n \in \mathbb{Z}_{\geq 0}$ (see [St07, Theorem 3.10] or [OT, Theorem 2.47]) and $|C(\mathcal{A})|$ is the sum of the coefficients $c_0, c_1, \ldots, c_n$.

2.5. **Characteristic polynomials and chromatic polynomials.** Another important aspect of characteristic polynomials is that they contain chromatic polynomials of graphs.

Let $\Gamma = (V, E)$ be a simple graph, that is, a pair of a finite set V and a family E of 2-elements subsets of V. An elements of V is called a **vertex** of Γ and an element of E is called an **edge** of Γ. A **proper k-coloring** of Γ is a map $c : V \to [k]$ satisfying $c(i) \neq c(j)$ for all $\{i, j\} \in E$. The **chromatic function** of Γ is the function $\chi_\Gamma : \mathbb{N} \to \mathbb{N}$ defined by

$$\chi_\Gamma(k) = \text{the number of proper } k\text{-colorings of } \Gamma.$$

As we see below a chromatic function is actually a polynomial and is nothing but a characteristic polynomial of an arrangement called a graphic arrangement.

Fix an integer n. For every $\{i, j\} \subset [n] = \{1, 2, \ldots, n\}$, let

$$H_{i,j} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i - x_j = 0\}.$$

The **graphic arrangement** of a simple graph $\Gamma = ([n], E)$, is the arrangement

$$\mathcal{A}_\Gamma = \{H_{i,j} \mid \{i, j\} \in E\}.$$

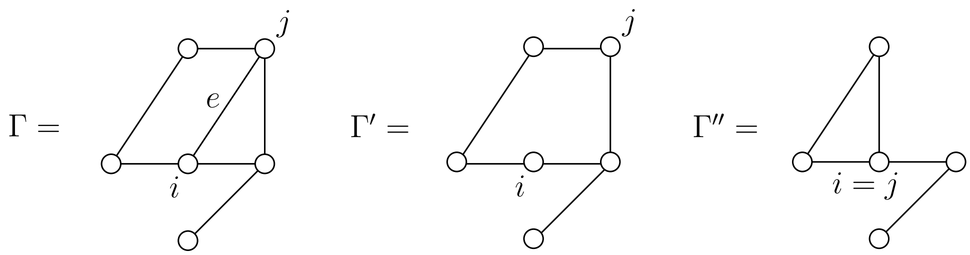


FIGURE 4. Examples of Γ, Γ' and Γ''.

The next result is well-known.

Theorem 2.7. *If* $\Gamma = ([n], E)$ *is a simple graph, then* $\chi_\Gamma(k) = \chi_{\mathcal{A}_\Gamma}(k)$ *for all* $k \in \mathbb{N}$.

Proof sketch. If Γ has no edges (that is, $E = \varnothing$) then it is clear that $\chi_\Gamma(k) = k^n = \chi_{\mathcal{A}_\Gamma}(k)$ for all $k \in \mathbb{N}$. Suppose that Γ has at least one edge.

Fix $e = \{i, j\} \in E$. Let Γ' be the graph obtained from Γ by removing the edge e and let Γ'' be the graph obtained from Γ by contracting an edge e (that is, the graph obtained from Γ by identifying vertices i and j). See Figure 4. Then it is easy to see that

$$\chi_\Gamma(k) = \chi_{\Gamma'}(k) - \chi_{\Gamma''}(k) \quad \text{for all } k \in \mathbb{N}. \tag{2.2}$$

Indeed, proper colorings of Γ are the proper colorings c of Γ' satisfying $c(i) \neq c(j)$, so $\chi_\Gamma(k)$ equals to $\chi_{\Gamma'}(k)$ minus "the number of proper k-colorings c of Γ' with $c(i) = c(j)$", which equals to the number of proper k-colorings of Γ''.

On the other hand, it is clear that $\mathcal{A}_{\Gamma'} = \mathcal{A} \setminus H_{i,j}$, and by taking an appropriate coordinate of $H_{i,j} \cong \mathbb{R}^{n-1}$ the arrangement $\mathcal{A}_{\Gamma''}$ is identified with $\mathcal{A}/H_{i,j}$. Then the deletion-restriction formula and (2.2) guarantee the desired statement. ☐

Theorem 2.7 sometimes allows us to compute characteristic polynomials quite easily. For example, consider the arrangement

$$\mathcal{A} = \{H_{i,j} \mid 1 \leq i < j \leq n\}$$

in $\mathbb{R}^n$. This arrangement is called the **braid arrangement**, and is the graphic arrangement of the complete graph K_n (i.e., the graph whose vertex set is $[n]$ and whose edge set is $\{\{i, j\} \subset [n] \mid 1 \leq i < j \leq n\}$). Then, since the number of proper k-colorings of K_n is

$$k \times (k-1) \times (k-2) \times \cdots \times (k-n+1),$$

we see by the previous theorem that the characteristic polynomial of $\mathcal{A}$ is given by

$$\chi_\mathcal{A}(q) = \chi_{K_n}(q) = q(q-1)\cdots(q-n+1).$$

2.6. **Poset of chambers and distance enumerator polynomials.** Let $\mathcal{A}$ be an arrangement in V. Fix a chamber $F \in C(\mathcal{A})$ (often called a base region). For each $R \in C(\mathcal{A})$, define

$$\operatorname{Sep}(R, F) := \{H \in \mathcal{A} \mid H \text{ separates } R \text{ and } F\} \quad \text{and} \quad \operatorname{dist}(R, F) := |\operatorname{Sep}(R, F)|.$$

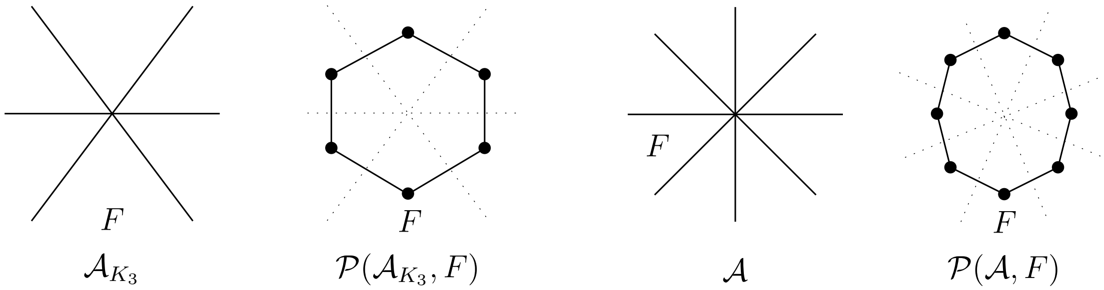


Figure 5. Hasse diagrams of posets of chambers

We define the partial order on $C(\mathcal{A})$ by

$$R \leq R' \quad \Leftrightarrow \quad \mathrm{Sep}(R, F) \subset \mathrm{Sep}(R', F), \tag{2.3}$$

and denote by $\mathcal{P}(\mathcal{A}, F)$ the poset whose elements are chambers of $\mathcal{A}$ and whose order is (2.3). The poset $\mathcal{P}(\mathcal{A}, F)$ has the unique smallest element F with $\mathrm{Sep}(F, F) = \varnothing$ and the unique largest element $-F$ with $\mathrm{Sep}(-F, F) = \mathcal{A}$. Also, it has a natural rank function defined by $\mathrm{rank}(R) = \mathrm{dist}(R, F)$. The rank generating function of $\mathcal{P}(\mathcal{A}, F)$ is called the **distance enumerator polynomial** of $\mathcal{A}$ w.r.t. F, and denoted by $\mathrm{Dist}_{\mathcal{A},F}(q)$. In other words,

$$\mathrm{Dist}_{\mathcal{A},F}(q) := \sum_{R \in C(\mathcal{A})} q^{\mathrm{dist}(R,F)}.$$

Example 2.8. In Figure 5, we write Hasse diagrams of the poset of chambers for the braid arrangement $\mathcal{A}_{K_3}$ and for the arrangement $\mathcal{A}$ in Figure 2 (the posets do not depend on the choice of $F \in C(\mathcal{A})$ for these cases). The distance enumerator polynomials for these arrangements are given by

$$\mathrm{Dist}_{\mathcal{A}_{K_3},F}(q) = 1 + 2q + 2q^2 + q^3$$

and

$$\mathrm{Dist}_{\mathcal{A},F}(q) = 1 + 2q + 2q^2 + 2q^3 + q^4.$$

A polynomial $\sum_{k=0}^{d} a_k q^k$, where $a_k \in \mathbb{Z}$ and $a_d \neq 0$, is said to be **palindromic** if $a_k = a_{d-k}$ for all k. Here are some basic properties of posets of chambers and distance enumerator polynomials (see [Ed] for the proofs).

Lemma 2.9. *Let $\mathcal{A}$ be an arrangement in V and $F \in C(\mathcal{A})$.*

1. *$\mathcal{P}(\mathcal{A}, F)$ is graded of rank $|\mathcal{A}|$, that is, for every maximal chain $F = R_0 \lneq R_1 \lneq \cdots \lneq R_l$ in $\mathcal{P}(\mathcal{A}, F)$ we have $l = |\mathcal{A}|$.*
2. *$\mathcal{P}(\mathcal{A}, F)$ is self-dual by the map $R \to -R$. In particular, $\mathrm{Dist}_{\mathcal{A},F}(q)$ is a palindromic polynomial of degree $|\mathcal{A}|$.*

3. Chambers of subarrangements of Weyl arrangements

The Betti numbers of regular Hessenberg varieties are closely related to the combinatorics of chambers of subarrangement of the Weyl arrangement. Indeed, as we saw in the introduction, Poincaré polynomials of regular Hessenberg varieties can

	Φ	elements of Δ	Φ^+
A_{n-1}	$\{\pm\mathbf{e}_i-\mathbf{e}_j \mid 1\leq i<j\leq n\}$	$\mathbf{e}_1-\mathbf{e}_2,\dots,\mathbf{e}_{n-1}-\mathbf{e}_n$	$\{\mathbf{e}_i-\mathbf{e}_j \mid 1\leq i<j\leq n\}$
B_n	$\{\pm\mathbf{e}_i \mid i=1,2,\dots,n\}$ $\cup\{\pm(\mathbf{e}_j-\mathbf{e}_i) \mid 1\leq i<j\leq n\}$ $\cup\{\pm(\mathbf{e}_j+\mathbf{e}_i) \mid 1\leq i<j\leq n\}$	$\mathbf{e}_1-\mathbf{e}_2,\dots,\mathbf{e}_{n-1}-\mathbf{e}_n,\mathbf{e}_n$	$\{\mathbf{e}_i \mid i=1,2,\dots,n\}$ $\cup\{\mathbf{e}_i-\mathbf{e}_j \mid 1\leq i<j\leq n\}$ $\cup\{\mathbf{e}_i+\mathbf{e}_j \mid 1\leq i<j\leq n\}$
C_n	$\{\pm 2\mathbf{e}_i \mid i=1,2,\dots,n\}$ $\cup\{\pm(\mathbf{e}_j-\mathbf{e}_i) \mid 1\leq i<j\leq n\}$ $\cup\{\pm(\mathbf{e}_j+\mathbf{e}_i) \mid 1\leq i<j\leq n\}$	$\mathbf{e}_1-\mathbf{e}_2,\dots,\mathbf{e}_{n-1}-\mathbf{e}_n,2\mathbf{e}_n$	$\{2\mathbf{e}_i \mid i=1,2,\dots,n\}$ $\cup\{\mathbf{e}_i-\mathbf{e}_j \mid 1\leq i<j\leq n\}$ $\cup\{\mathbf{e}_i+\mathbf{e}_j \mid 1\leq i<j\leq n\}$
D_n	$\{\pm(\mathbf{e}_j-\mathbf{e}_i) \mid 1\leq i<j\leq n\}$ $\cup\{\pm(\mathbf{e}_j+\mathbf{e}_i) \mid 1\leq i<j\leq n\}$	$\mathbf{e}_1-\mathbf{e}_2,\dots,\mathbf{e}_{n-1}-\mathbf{e}_n,\mathbf{e}_{n-1}+\mathbf{e}_n$	$\{\mathbf{e}_i-\mathbf{e}_j \mid 1\leq i<j\leq n\}$ $\cup\{\mathbf{e}_i+\mathbf{e}_j \mid 1\leq i<j\leq n\}$

TABLE 1. Root systems. For type A_{n-1}, we consider that Φ is a subset of $\{(a_1,\dots,a_n)\in\mathbb{R}^n \mid a_1+\cdots+a_n=0\}\cong\mathbb{R}^{n-1}$.

be considered as a certain generating function of chambers. In this section, we will explain this connection for regular nilpotent and regular semisimple Hessenberg varieties.

3.1. **Root systems and Weyl groups.** We first review some basic facts on root systems and Weyl groups. Here we only introduce basic combinatorial properties and refer the readers to [Hum1, BB] for more information.

Let $V\cong\mathbb{R}^n$ be an n-dimensional Euclidian space with an inner product $(-,-)$. For a non-zero element $\alpha\in V$, let H_α be the linear hyperplane orthogonal to α and let s_α be the reflection defined by the hyperplane H_α. A finite subset $\Phi\subset V$ is said to be a **root system** if the following conditions are satisfied:

(R1) Φ spans V and $\mathbf{0}\notin\Phi$;
(R2) If $\alpha\in\Phi$, then the only multiples of α in Φ are $\pm\alpha$;
(R3) If $\alpha\in\Phi$, then $s_\alpha(\Phi)=\Phi$;
(R4) If $\alpha,\alpha'\in\Phi$ then $2(\alpha,\alpha')/(\alpha,\alpha)\in\mathbb{Z}$.

An element of a root system is called a **root**. If $\Phi\subset V$ is a root system, then there is a base $\Delta=\{\alpha_1,\dots,\alpha_n\}\subset\Phi$ of V such that every element $\alpha\in\Phi$ is contained in either $\mathbb{Z}_{\geq 0}\alpha_1+\cdots+\mathbb{Z}_{\geq 0}\alpha_n$ or $\mathbb{Z}_{\leq 0}\alpha_1+\cdots+\mathbb{Z}_{\leq 0}\alpha_n$. Such a set Δ is called a **base** for Φ and elements of Δ are called **simple roots**. A root α in $\sum_{k=1}^n\mathbb{Z}_{\geq 0}\alpha_k$ is called a **positive root** and a root α in $\sum_{k=1}^n\mathbb{Z}_{\leq 0}\alpha_k$ is called a **negative root**. Given a basis Δ, we denote by Φ^+ and Φ^- the set of positive roots and the set of negative roots respectively. We say that a root system Φ is **irreducible** if it cannot be partitioned into the union of two proper subset such that each root in one set is orthogonal to each root in the other set.

If you are not familiar with root systems, you may just understand the following: Irreducible root systems are completely classified and there are four infinite families called type A_n, B_n, C_n, D_n and finite exceptional types E_6, E_7, E_8, F_4 and G_2. For each type, there is a unique choice of Φ and Δ up to isomorphism. Table 1 list one possible choice of (Φ,Φ^+,Δ) for types A_n, B_n, C_n and D_n. We basically use this choice throughout this chapter, and write $\Phi_{A_n},\Phi_{B_n},\Phi_{C_n},\Phi_{D_n}$ for these root system of type A_n, B_n, C_n and D_n respectively.

In the rest of this section, we fix a root system Φ and its basis $\Delta = \{\alpha_1, \dots, \alpha_n\}$. The **Weyl group** $W = W_\Phi$ of Φ is the group generated by the reflections $\{s_\alpha \mid \alpha \in \Phi^+\}$. Let $s_1, \dots, s_n$ be reflections defined by hyperplanes $H_{\alpha_1}, \dots, H_{\alpha_n}$ respectively. The Weyl group is known to be generated by $s_1, \dots, s_n$, so every element $w \in W$ can be written as a product of $s_1, \dots, s_n$. The **length** of $w \in W$ is

$$\ell(w) := \min\{k \mid w = s_{i_1} \cdots s_{i_k} \text{ for some } i_1, \dots, i_k \in [n]\}.$$

The **left weak Bruhat order** on W is the partial order on W defined by

$$w \succ w' \quad \Leftrightarrow \quad w = s_{i_1} \cdots s_{i_k} w' \text{ for some } i_1, \dots, i_k \in [n] \text{ with } \ell(w) = \ell(w') + k.$$

Below, we note a few basic properties of the left weak Bruhat order which we will need. For $w \in W$, we define the set $N(w)$ by

$$N(w) := \{\alpha \in \Phi^+ \mid w(\alpha) \in \Phi^-\}.$$

Lemma 3.1. *Let $w, w' \in W$ and $\alpha \in \Delta$. Then*

(1) *$w \succeq w'$ if and only if $N(w) \supset N(w')$.*
(2) *$\ell(w) = |N(w)|$.*
(3) *$\ell(s_\alpha w) = \ell(w) + 1$ if $w^{-1}(\alpha) \in \Phi^+$ and $\ell(s_\alpha w) = \ell(w) - 1$ if $w^{-1}(\alpha) \in \Phi^-$.*

We explain combinatorial meanings of the above lemma in terms of chambers of Weyl arrangements in the next subsection, and since we will only use that combinatorial aspect in this chapter we omit the proofs of the above lemma and refer the readers to [BB, §3.1] and [Hum2, §1.6 and §1.7].

3.2. **Weyl arrangements.** The **Weyl arrangement** $\mathcal{A}_{\Phi^+}$ for a root system Φ is the arrangement

$$\mathcal{A}_{\Phi^+} := \{H_\alpha \mid \alpha \in \Phi^+\}.$$

(We use Φ^+ instead of Φ simply because $H_\alpha = H_{-\alpha}$.) We will explain that Weyl arrangements give a nice geometric model to visualize Weyl groups and their weak Bruhat order.

A chambers of $\mathcal{A}_{\Phi^+}$ is called a **Weyl chamber**, and the polyhedral cone

$$F = \bigcap_{\alpha \in \Delta} \{\boldsymbol{x} \mid (\alpha, \boldsymbol{x}) < 0\} = \bigcap_{\alpha \in \Phi^+} \{\boldsymbol{x} \mid (\alpha, \boldsymbol{x}) < 0\},$$

which is a chamber of $\mathcal{A}_{\Phi^+}$, is called the **fundamental Weyl chamber**. It is well-known that the Weyl group W acts on $C(\mathcal{A}_{\Phi^+})$ transitively, so, the correspondence[4]

$$W \ni w \longmapsto C_w := w^{-1}(F) \in C(\mathcal{A}_{\Phi^+})$$

gives a bijection from W to the set of chambers of $\mathcal{A}_{\Phi^+}$.

Example 3.2. If $\Phi = \Phi_{A_{n-1}}$, then the arrangement $\mathcal{A}_{\Phi^+}$ can be identified with the braid arrangement $\mathcal{A}_{K_n}$ and its Weyl group W is isomorphic to the nth symmetric group S_n. The fundamental chamber is the chamber that contains the point $(1, 2, \dots, n)$. If we denote by $C_{(a_1, \dots, a_n)}$ the chamber that contains the point $(a_1, \dots, a_n)$, then

$$C(\mathcal{A}_{\Phi^+}) = \{C_{(a_1, \dots, a_n)} \mid \{a_1, \dots, a_n\} = \{1, 2, \dots, n\}\}$$

[4]It might be more common to consider the correspondence $w \to w(F)$, but considering $w^{-1}(F)$ is more convenient in our setting.

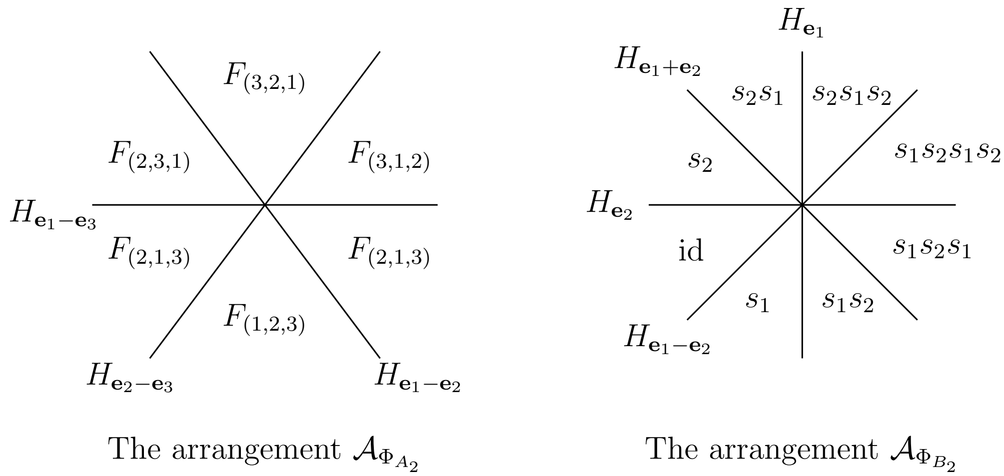


FIGURE 6. Chambers of Weyl arrangements. One should be a little careful that we are setting $C_w = w^{-1}(F)$.

and the bijection from S_n to $C(\mathcal{A}_{\Phi^+})$ is given by

$$S_n \ni \sigma \to C_{(\sigma(1),\dots,\sigma(n))} \in C(\mathcal{A}_{\Phi^+}).$$

Figure 6 illustrate a picture for the case when $n = 3$.

Example 3.3. The arrangement $\mathcal{A}_{\Phi^+_{B_2}}$ consists of four hyperplanes defined by equations

$$x_1 = 0, \quad x_2 = 0, \quad x_1 - x_2 = 0, \quad x_1 + x_2 = 0,$$

so it is nothing but the arrangement given in Figure 2. In this case, the Weyl group W is the dihedral group of order 8 and its fundamental chamber is a region defined by $x_1 - x_2 < 0$ and $x_2 < 0$. The Weyl group W is generated by reflections s_1 and s_2 defined by the hyperplanes $H_{\alpha_1} = H_{\mathbf{e}_1-\mathbf{e}_2}$ and $H_{\alpha_2} = H_{\mathbf{e}_2}$ respectively, and consists of 8 elements

$$\mathrm{id}, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1, s_2s_1s_2, s_1s_2s_1s_2 = s_2s_1s_2s_1.$$

The correspondence between these elements and chambers of $\mathcal{A}_{\Phi^+_{B_2}}$ is illustrated in Figure 6.

In the previous section, we define the poset of chambers of arrangements. The poset of the weak Bruhat order is known to be isomorphic to the poset $\mathcal{P}(\mathcal{A}_{\Phi^+}, F)$. Indeed, we have the following lemma.

Lemma 3.4. *For any $w \in W$, one has* $\mathrm{Sep}(C_w, F) = \{H_\alpha \mid \alpha \in N(w)\}$.

Proof. Take a point $\mathbf{p} \in F$ and $\alpha \in \Phi$. Then by the definition of F one has $\alpha \in \Phi^-$ if and only if $(\alpha, \mathbf{p}) > 0$. Thus, for $\alpha \in \Phi^+$, we have $\alpha \in N(w)$ if and only if $(w(\alpha), \mathbf{p}) > 0$, which is equivalent to $(\alpha, w^{-1}(\mathbf{p})) > 0$. The last condition is equivalent to saying that H_α separates F and $C_w = w^{-1}(F)$ since $(\alpha, \mathbf{p}) < 0$. ☐

To simplify notation, we write $\mathrm{Sep}(C_w, F) = \mathrm{Sep}(C_w)$. The above lemma tells that we may identify $N(w)$ and $\mathrm{Sep}(C_w)$. Using this Lemma 3.1(1) and (2) can be stated in the following form.

Corollary 3.5. *Let $w, w' \in W$. Then we have*

(1) $w \succeq w'$ *if and only if* $\mathrm{Sep}(C_w) \supset \mathrm{Sep}(C_{w'})$.
(2) $\ell(w) = \mathrm{dist}(C_w, F)$.

For a chamber $R \in C(\mathcal{A}_{\Phi^+})$, let $\mathrm{Supp}(R)$ be the set of hyperplanes $H_\alpha \in \mathcal{A}_{\Phi^+}$ which is a supporting hyperplane of a facet of the closure $\overline{R}$ of R. The next statement gives a geometric meaning of Lemma 3.1(3).

Lemma 3.6. *Let $w \in W$ and $\alpha \in \Delta$. Then we have*

(1) $\mathrm{Supp}(C_w) = \{H_\gamma \mid \gamma \in w^{-1}(\Delta)\}$.
(2) *If* $w^{-1}(\alpha) \in \Phi^-$ *then* $\mathrm{Sep}(C_{s_\alpha w}) = \mathrm{Sep}(C_w) \setminus \{H_{w^{-1}(\alpha)}\}$.
(3) *If* $w^{-1}(\alpha) \in \Phi^+$ *then* $\mathrm{Sep}(C_{s_\alpha w}) = \mathrm{Sep}(C_w) \cup \{H_{w^{-1}(\alpha)}\}$.

Proof. By the definition of F, we have $\mathrm{Supp}(F) = \{H_\gamma \mid \gamma \in \Delta\}$, so we have

$$\mathrm{Supp}(w^{-1}(F)) = \{w^{-1}(H_\alpha) \mid \alpha \in \Delta\} = \{H_{w^{-1}(\alpha)} \mid \alpha \in \Delta\} = \{H_\alpha \mid \alpha \in w^{-1}(\Delta)\},$$

proving (1). The statement (1) tells that $\mathrm{Sep}(s_{w^{-1}(\alpha)}(C_w))$ is $\mathrm{Sep}(C_w) \setminus \{H_{w^{-1}(\alpha)}\}$ if $w^{-1}(\alpha) \in \Phi^-$ and is $\mathrm{Sep}(C_w) \cup \{H_{w^{-1}(\alpha)}\}$ if $w^{-1}(\alpha) \in \Phi^+$. Then statements (2) and (3) follow from the following computation

$$s_{w^{-1}(\alpha)}(C_w) = s_{w^{-1}(\alpha)}(w^{-1}(F)) = w^{-1}s_\alpha(F) = (s_\alpha w)^{-1}(F) = C_{s_\alpha w},$$

where we use $s_{w^{-1}(\alpha)} = w^{-1}s_\alpha w$ for the second equality. □

3.3. **Poincaré polynomials of regular Hessenberg varieties.** Here we explain that using a relation between $N(w)$ and $\mathrm{Sep}(C_w)$ in the previous subsection we can visualize Poincaré polynomials of regular semisimple Hessenberg varieties.

We first recall the definition of Hessenberg varieties. We note that our definition uses lower ideals instead of Hessenberg functions, so it is slightly different to those given in previous chapters, but for type A case it coincides with the definition of Hessenberg varieties in terms of Hessenberg functions as we will explain soon. Let G be a simple complex linear algebraic groupof rank n, B a Borel subgroup of G, $T \subset B$ a maximal torus, and $W = N(T)/T$ its Weyl group. We write $\mathfrak{g}$ and $\mathfrak{b}$ for the corresponding linear algebraic group of G and B, respectively. Let Φ, Φ^+, Φ^-, and $\Delta = \{\alpha_1, \dots, \alpha_n\}$ be the set of roots, the set of positive roots, the set of negative roots and the set of simple roots associated with these data. Recall that every positive root $\alpha \in \Phi^+$ can be written as a non-negative linear combination of simple roots $\alpha_1, \dots, \alpha_n$. The set Φ^+ of positive roots has a natural poset structure defined by $\alpha \succ \beta$ if $\alpha - \beta \in \sum_{k=1}^n \mathbb{Z}_{\geq 0}\alpha_k$. See Figure 7. A **lower ideal** of Φ^+ is a subset of $I \subset \Phi^+$ satisfying the condition that for any $\alpha \in I$ and $\beta \in \Phi^+$ with $\beta \preceq \alpha$ one has $\beta \in I$.

Example 3.7. Hessenberg functions of length n can be identified with lower ideals in $\Phi^+_{A_{n-1}}$. For a Hessenberg function $h = (h(1), \dots, h(n))$, that is, a non-decreasing sequence of integers satisfying $k \leq h(k) \leq n$ for all k, we define $I_h \subset \Phi^+_{A_{n-1}}$ by

$$I_h = \{\mathbf{e}_i - \mathbf{e}_j \mid i + 1 \leq j \leq h(i)\}.$$

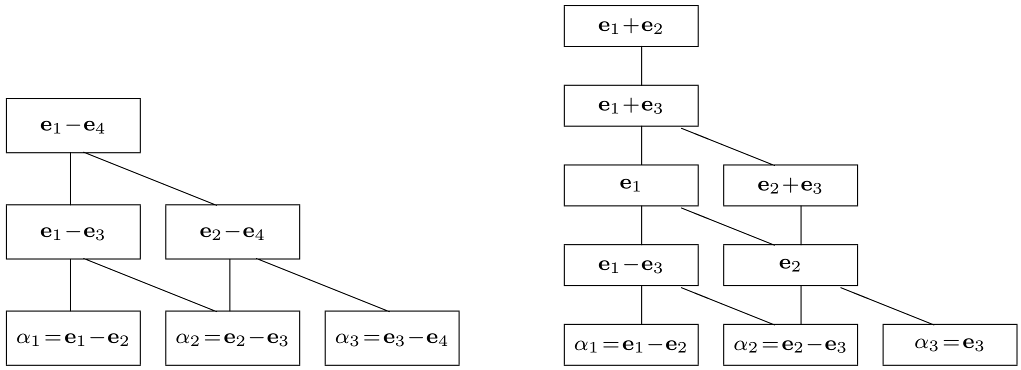


FIGURE 7. Root posets for type A_3 and B_3

This set I_h becomes a lower ideal of $\Phi^+_{A_{n-1}}$. Conversely, given a lower ideal $I \subset \Phi^+_{A_{n-1}}$, if we define the function $h_I = (h(1), \dots, h(n))$ by

$$h(k) = |\{\mathbf{e}_k - \mathbf{e}_j \mid k < j \leq n\} \cap I| + k \quad (\text{for } k = 1, 2, \dots, n),$$

then h_I becomes a Hessenberg function. These constructions $h \to I_h$ and $I \to h_I$ give a one to one correspondence between lower ideals in $\Phi^+_{A_{n-1}}$ and Hessenberg functions of length n.

For a lower ideal $I \subset \Phi^+$, its **Hessenberg space** is $H(I) = \mathfrak{b} \bigoplus_{\alpha \in I} \mathfrak{g}_{-\alpha}$, where $\mathfrak{g}_\gamma$ is the root space for $\gamma \in \Phi$. For $X \in \mathfrak{g}$ and an ideal $I \subset \Phi^+$, the Hessenberg variety $\mathrm{Hess}(X, I)$ is defined by

$$\mathrm{Hess}(X, I) = \{gB \in G/B \mid \mathrm{Ad}(g^{-1}) \cdot X \in H(I)\},$$

where $\mathrm{Ad}(-)$ is the adjoint action. We note that, by the correspondence given in Example 3.7, in type A_{n-1}, the above definition of Hesseberg variety $\mathrm{Hess}(X, I)$ coincide with the definition of the Hessenberg variety associate with the matrix X and the Hessenberg function h_I. See also Definition 2.8 in Chapter 8 for the equivalence of these definitions. For a regular nilpotent element[5] $\mathsf{N} \in \mathfrak{g}$ (resp. regular semisimple element $\mathsf{S} \in \mathfrak{g}$), the Hessenberg variety $\mathrm{Hess}(\mathsf{N}, I)$ (resp. $\mathrm{Hess}(\mathsf{S}, I)$) is called the **regular nilpotent** (resp. **regular semisimple**) Hessenberg variety for I.

As we mentioned in the beginning of this section, there is a nice combinatorial formula of Poincaré polynomials of regular Hessenberg varieties. For regular semisimple and regular nilpotent Hessenberg varieties, this formula is given in the following form.

Theorem 3.8 (De Mari-Procesi-Shayman [dMPS]). *Let* $\mathrm{Hess}(\mathsf{S}, I)$ *be the regular semisimple Hessenberg variety for a lower ideal* I. *Then*

$$\mathrm{Poin}\big(\mathrm{Hess}(\mathsf{S}, I), q\big) = \sum_{w \in W} q^{2|N(w) \cap I|}.$$

[5]We may take $\mathsf{N} \in \mathfrak{g}$ as the sum of simple roots, that is, the sum of a basis of $\mathfrak{g}_\alpha$ with $\alpha \in \Delta$.

Theorem 3.9 (Tymoczko [Ty07], Precup [Pr13])**.** *Let* $\mathrm{Hess}(\mathsf{N}, I)$ *be the regular nilpotent Hessenberg variety for a lower ideal* I*. Then*

$$\mathrm{Poin}\big(\mathrm{Hess}(\mathsf{N}, I), q\big) = \sum_{w \in W,\ w^{-1}(\Delta) \subset (-I) \cup \Phi^+} q^{2|N(w) \cap I|}.$$

We will not prove these results, but will explain an arrangement theoretic meaning of the RHS of the above formulas.

For a lower ideal $I \subset \Phi^+$, the arrangement $\mathcal{A}_I = \{H_\alpha \mid \alpha \in I\}$ is called the **ideal arrangement** associated with I. For two Weyl chambers $R, R' \in C(\mathcal{A}_{\Phi^+})$, we define

$$\mathrm{dist}_I(R, R') = |\mathrm{Sep}(R, R') \cap \mathcal{A}_I|.$$

Thus $\mathrm{dist}_I(R, R')$ is the number of hyperplanes in $\mathcal{A}_I$ that separates R and R'. Then, since $\mathrm{Sep}(C_w, F)$ is naturally identified with $N(w)$ by Lemma 3.4, the above formula of Poincaré polynomials can be rewritten in the following form.

Theorem 3.10. *For a lower ideal* $I \subset \Phi^+$*, one has*

$$\mathrm{Poin}\big(\mathrm{Hess}(\mathsf{S}, I), \sqrt{q}\big) = \textstyle\sum_{R \in C(\mathcal{A}_{\Phi^+})} q^{\mathrm{dist}_I(R, F)}.$$

The above formula is, of course, just an easy reformulation of Theorem 3.8. But it certainly gives a way to visualize the Poincaré series geometrically. For example, if we consider the lower ideal $I = \{\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_2, \mathbf{e}_1\}$ of $\Phi^+_{B_2}$, then the Poincaré polynomial $\mathrm{Poin}(\mathrm{Hess}(\mathsf{S}, I), q)$ is $1 + 3q^2 + 3q^4 + q^6$, which can be considered as a generating function of Weyl chambers in terms of $\mathrm{dist}_I(-, F)$ as illustrated in Figure 8.

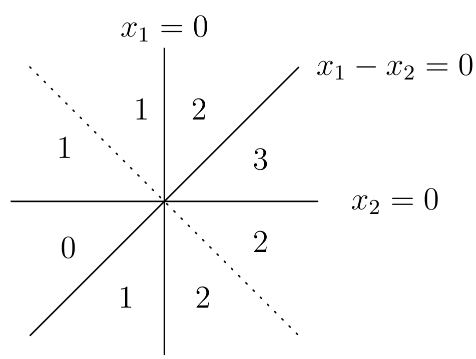


FIGURE 8. Visualization of the Poincaré polynomial of $\mathrm{Hess}(\mathsf{S}, I)$. Numbers on each Weyl chamber R is the distances $\mathrm{dist}_I(R, F)$.

3.4. **Chambers of subarrangements of Weyl arrangements.** Our next goal is to explain an arrangement theoretic meaning of the Poincaré polynomial of $\mathrm{Hess}(\mathsf{N}, I)$ in Theorem 3.9. To do this, we study chambers of subarrangements of the Weyl arrangement $\mathcal{A}_{\Phi^+}$. For $J \subset \Phi^+$, we define the subarrangement $\mathcal{A}_J$ of $\mathcal{A}_{\Phi^+}$ by

$$\mathcal{A}_J := \{H_\alpha \mid \alpha \in J\}.$$

Since $\mathcal{A}_J$ is a subarrangement of $\mathcal{A}_{\Phi^+}$ each chamber of $\mathcal{A}_J$ is essentially a union of some Weyl chambers, that is, for each $R \in C(\mathcal{A}_J)$ there are $w_1, \ldots, w_t \in W$ such

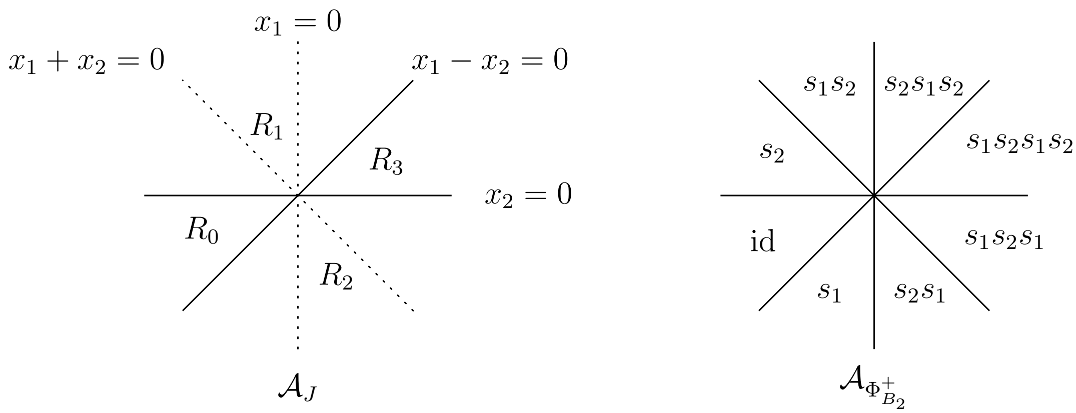


FIGURE 9. Chambers of $\mathcal{A}_J$ when $J = \{\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_2\}$.

that $\overline{R} = \overline{C_{w_1}} \cup \cdots \cup \overline{C_{w_t}}$. We write $U(R)$ for the set of these elements $w_1, \dots, w_t$, that is,

$$U(R) := \{w \in W \mid C_w \subset R\}.$$

Example 3.11. Let $J = \{\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_2\} \subset \Phi^+_{B_2}$. Then $\mathcal{A}_J$ has four chambers R_0, R_1, R_2, R_3 (see Figure 9). Comparing chambers of $\mathcal{A}_{\Phi^+_{B_2}}$ we have

$$\begin{aligned} U(R_0) &= \{\mathrm{id}\}, \\ U(R_1) &= \{s_2, s_1 s_2, s_2 s_1 s_2\}, \\ U(R_2) &= \{s_1, s_2 s_1, s_1 s_2 s_1\}, \\ U(R_3) &= \{s_1 s_2 s_1 s_2\}. \end{aligned}$$

We are interested in subarrangements $\mathcal{A}_J$ such that $U(R)$ contains exactly one smallest element w.r.t. the left weak Bruhat order for each $R \in C(\mathcal{A}_J)$. Our goal is to prove the following statement. A subset J of Φ^+ is said to be **coconvex** if it satisfies the following condition

$(\star)$ if both $\alpha, \beta \in \Phi^+$ are not in J then $\alpha + \beta$ is also not in J.

Theorem 3.12. *If $J \subset \Phi^+$ is coconvex, then, for every chamber $R \in C(\mathcal{A}_J)$, the set $U(R)$ has the unique smallest element w.r.t. the left weak Bruhat order.*

We actually prove a bit stronger statement. For $w, w' \in W$ and $\alpha \in \Delta$, we write $w \xrightarrow{\alpha} w'$ if $w = s_\alpha w'$ and $w \succ w'$ (in this case w covers w' by Lemma 3.6). Let Γ_{Φ^+} be the directed graph such that

(1) the vertex set of Γ_{Φ^+} is W, and
(2) we have an arrow $w \to w'$ in Γ_{Φ^+} if $w \xrightarrow{\alpha} w'$ for some $\alpha \in \Delta$.

Thus Γ_{Φ^+} is the digraph of the Hesse diagram of the poset of the left weak Bruhat order. When $w \xrightarrow{\alpha} w'$, we call α a **label** of an edge $w \to w'$ in the graph Γ_{Φ^+}. It is convenient to consider this graph as a graph whose vertices are Weyl chambers and whose arrows are arrows between two adjacent chambers. From this point of view, we have $w \to w'$ if and only if $\overline{C}_w$ and $\overline{C}_{w'}$ have a common facet whose

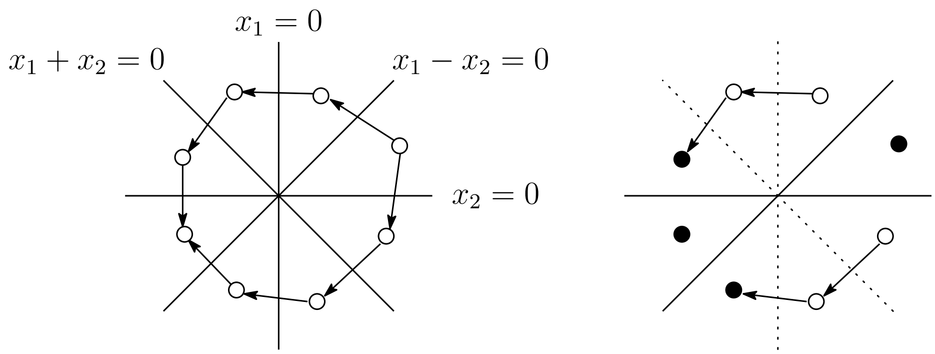


FIGURE 10. Directed graphs $\Gamma_{\Phi^+_{B_2}}$ and Γ_J for $J = \{\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_2\}$. The graph Γ_J has 4 connected components, each corresponds to a chamber of $\mathcal{A}_J$ and has the unique sink represented by a black circle.

supporting hyperplane separates F and C_w. Also, the label of an arrow $w \to w'$ is the root α such that $H_{w^{-1}(\alpha)}$ is the hyperplane that the arrow cross (i.e., $\{H_{w^{-1}(\alpha)}\} = \mathrm{Supp}(C_w) \cap \mathrm{Supp}(C_{w'})$), and we have

$$w \xrightarrow{\alpha} w' \iff \mathrm{Sep}(C_w) = \mathrm{Sep}(C_{w'}) \cup \{H_{w^{-1}(\alpha)}\}.$$

(Recall that $\mathrm{Supp}(C_w) = \{H_{w^{-1}(\alpha)} \mid \alpha \in \Delta\}$.)

For $J \subset \Phi^+$, let Γ_J be the subgraph of Γ_{Φ^+} such that the vertex set of Γ_J is W and arrows of Γ_J are arrows $w \to w'$ of Γ_{Φ^+} whose label α satisfies $H_{w^{-1}(\alpha)} \notin \mathcal{A}_J$ (or, $w^{-1}(\alpha) \notin J$). Thus Γ_J is the graph obtained from Γ_{Φ^+} by removing all edges that cross a hyperplane in $\mathcal{A}_J$. See Figure 10. If we regard Γ_J as an undirected graph, then two vertices $w, w' \in W$ are connected if C_w and $C_{w'}$ are contained in the same chamber of $\mathcal{A}_J$. For a chamber $R \in C(\mathcal{A}_J)$, let Γ_R be the connected component of Γ_J corresponding to $R \in C(\mathcal{A}_J)$. In other words, Γ_R is the induced subgraph of Γ_J on $U(R)$.

We call an element $w \in W$ a **sink** of Γ_J if Γ_J has no arrow $w \to w'$ with $w' \in W$. Thus, for each $R \in C(\mathcal{A}_J)$, an element $w \in U(R)$ is a sink if and only if it is a minimal element of $U(R)$ with respect to the left weak Bruhat order. We write $\mathrm{Sink}(J)$ for the set of all sinks of Γ_J. By Lemma 3.6, w is a sink of Γ_J if and only if for any $\alpha \in \Delta$ with $w^{-1}(\alpha) \in \Phi^-$ one has $H_{w^{-1}(\alpha)} \in \mathcal{A}_J$, equivalently, $w^{-1}(\Delta) \subset (-J) \cup \Phi^+$. In particular, we have

$$\mathrm{Sink}(J) = \{w \in W \mid w^{-1}(\Delta) \subset (-J) \cup \Phi^+\}.$$

(This shows that the right hand side of Theorem 3.9 is a certain counting function of $\mathrm{Sink}(J)$.) We will prove the following statement which implies Theorem 3.12.

Theorem 3.13. *Let J be a coconvex subset of Φ^+. Then each connected component Γ_R of Γ_J has the unique sink.*

For $w, w' \in W$, a **directed path** from w to w' in Γ_J is a sequence of arrows

$$w = w_0 \to w_1 \to w_2 \to \cdots \to w_p = w'$$

in Γ_J. If there is a directed path from w to w' in Γ_J, we write $w \to_J w'$.

Lemma 3.14. *Let $J \subset \Phi^+$ be a coconvex subset, and let $w_1, w_2 \in W$. If there is $w \in W$ such that $w \to w_1$ and $w \to w_2$ are arrows in Γ_J, then there is an $u \in W$ such that $w_1 \to_J u$ and $w_2 \to_J u$.*

Proof. By the assumption, there are $\alpha, \alpha' \in \Delta$ such that

(a) $w_1 = s_\alpha w$ and $w_2 = s_{\alpha'} w$,
(b) $H_{w^{-1}(\alpha)}, H_{w^{-1}(\alpha')}$ separate F and C_w, and
(c) $H_{w^{-1}(\alpha)}, H_{w^{-1}(\alpha')} \notin \mathcal{A}_J$.

Set

$$\Phi' = \Phi \cap \mathrm{span}_{\mathbb{Z}}\{\alpha, \alpha'\}.$$

Since α and α' are simple roots, Φ' is a root system (on $\mathrm{span}_{\mathbb{R}}\{\alpha', \alpha'\}$) with a basis $\Delta' = \{\alpha, \alpha'\}$, and the subgroup W' of W generated by $s_\alpha, s_{\alpha'}$ is a reflection group (see [Hum2, §1.10 Proposition(a)]). Since any root in $\Phi' \cap \Phi^+$ can be written as a non-negative linear combination of α and α', the conditions (b), (c) and ($\star$) tell

(b′) $H_{w^{-1}(\gamma)} \in \mathrm{Sep}(C_w)$ for any $\gamma \in \Phi' \cap \Phi^+$, and
(c′) $H_{w^{-1}(\gamma)} \notin \mathcal{A}_J$ for any $\gamma \in \Phi' \cap \Phi^+$.

Let $v_0 \in W'$ be the largest element of W'. Such an element must satisfy $\mathrm{Sep}(C_{v_0}) = \{H_\gamma \mid \gamma \in \Phi' \cap \Phi^+\}$. Thus H_γ separates $C_{v_0} = v_0^{-1}(F)$ and F for any $\gamma \in \Phi' \cap \Phi^+$. This in particular means that $H_{w^{-1}(\gamma)}$ separates $C_{v_0 w} = w^{-1}v_0^{-1}(F)$ and $C_w = w^{-1}(F)$ for any $\gamma \in \Phi' \cap \Phi^+$, so by (b′) we have

(d) $H_{w^{-1}(\gamma)} \notin \mathrm{Sep}(C_{v_0 w})$ for any $\gamma \in \Phi' \cap \Phi^+$.

Also, there are sequences of elements

$$v_1 = s_\alpha \prec v_2 \prec \cdots \prec v_p = v_0 \text{ and } v_1' = s_{\alpha'} \prec v_2' \prec \cdots \prec v_p' = v_0$$

such that $v_{i+1} = s_{\gamma_i} v_i$ and $v_{i+1}' = s_{\gamma_i'} v_i'$ with $\gamma_i, \gamma_i' \in \{\alpha, \alpha'\}$ for each i[6], where $p = |\Phi' \cap \Phi^+|$. We claim that Γ_J has the following two paths from w to $v_0 w$

$$w \to w_1 = v_1 w \to v_2 w \to \cdots \to v_{p-1} w \to v_p w = v_0 w \tag{3.1}$$

and

$$w \to w_2 = v_1' w \to v_2' w \to \cdots \to v_{p-1}' w \to v_p' w = v_0 w. \tag{3.2}$$

Note that the existence of the paths prove the desired statement by setting $u = v_0 w$.

We only prove the existence of the path (3.1) (the proof for the existence of (3.2) is similar). By Lemma 3.6, for each $i = 0, 1, \ldots, p-1$, there is exactly one hyperplane in $\mathcal{A}_{\Phi^+}$ that separates $C_{v_i w}$ and $C_{v_{i+1} w}$, namely

$$\mathrm{Sep}(C_{v_i w}) = \mathrm{Sep}(C_{v_{i+1} w}) \cup \{H\} \quad \text{or} \quad \mathrm{Sep}(C_{v_i w}) = \mathrm{Sep}(C_{v_{i+1} w}) \setminus \{H\} \tag{3.3}$$

for some hyperplane H. This in particular implies that there are at most $p = |\Phi' \cap \Phi^+|$ hyperplanes in $\mathcal{A}_{\Phi^+}$ that separate C_w and $C_{v_0 w}$. Then, since any hyperplane $H \in \mathrm{Sep}(C_w) \setminus \mathrm{Sep}(C_{v_0 w})$ separates C_w and $C_{v_0 w}$, it follows from (b′) and (d) that

$$\mathrm{Sep}(C_w) = \mathrm{Sep}(C_{v_0 w}) \cup \{H_{w^{-1}(\gamma)} \mid \gamma \in \Phi' \cap \Phi^+\}.$$

[6] more precisely $\gamma_i = \alpha$ if i is even and $\gamma_i = \alpha'$ is i is odd. Also, $\gamma_i' = \alpha'$ if i is even and $\gamma_i' = \alpha$ if i is odd.

This and (3.3) tell that $\mathrm{Sep}(C_{v_i w}) = \mathrm{Sep}(C_{v_{i+1} w}) \cup \{H_{w^{-1}\gamma}\}$ for some $\gamma \in \Phi' \cap \Phi^+$, so $v_i w \to v_{i+1} w$ is indeed a path of Γ_J. □

We now prove Theorem 3.13.

Proof of Theorem 3.13. Fix $R \in C(\mathcal{A}_J)$ and take two distinct elements $w, w' \in U(R)$. It suffices to prove that there is an element $w'' \in U(R)$ such that $w \to_J w''$ and $w' \to_J w''$. Since the graph Γ_R is connected as an undirected graph, there is a sequence of elements in $U(R)$

$$w = w_0, w_1, w_2, \ldots, w_m = w' \tag{3.4}$$

such that for each i we have $w_i \to w_{i+1}$ or $w_i \leftarrow w_{i+1}$ in Γ_J. If there is a j such that $w_{j-1} \leftarrow w_j \to w_{j+1}$, then by Lemma 3.14 there exists a path of the form

$$w_{j-1} \to \cdots \to u \leftarrow \cdots \leftarrow w_{j+1}$$

in Γ_J. This tells that we may take (3.4) so that there are no number j with $w_{j-1} \leftarrow w_j \to w_{j+1}$. Such a path must be a path of the form

$$w = w_0 \to \cdots \to w_k \leftarrow \cdots \leftarrow w_m = w'$$

for some k and the element w_k satisfies the desired condition. □

3.5. **Betti numbers of regular nilpotent Hessenberg varieties.** Recall that a subarrangement $\mathcal{A}_I$ of $\mathcal{A}_{\Phi^+}$ is called an **ideal arrangement** if I is a lower ideal of Φ^+. We now come to our goal to explain a connection between distance enumerator polynomials of ideal arrangements and Poincaré polynomials of regular nilpotent Hessenberg varieties. As we introduced in Theorem 3.9, the Poincaré polynomial of a regular nilpotent Hessenberg variety $\mathrm{Hess}(\mathsf{N}, I)$ is given by the following form

$$\mathrm{Poin}\big(\mathrm{Hess}(\mathsf{N}, I), q\big) = \sum_{w \in W,\ w^{-1}(\Delta) \subset (-I) \cup \Phi^+} q^{2|N(w) \cap I|}.$$

If I is a lower ideal, then I satisfies the coconvex condition ($\star$) in the previous subsection. Indeed, if I is a lower ideal, then, for $\alpha, \beta \in \Phi^+$, we have $\alpha + \beta \notin I$ when $\alpha \notin I$ or $\beta \notin I$. Then by using Theorem 3.13 the above formula of Poincaré polynomial can be rewritten in the following form.

Theorem 3.15 (Sommers-Tymoczko)**.** *Let I be a lower ideal of Φ^+ and let $R_0 \in C(\mathcal{A}_I)$ be the chamber of $\mathcal{A}_I$ that contains the fundamental Weyl chamber $F \in C(\mathcal{A}_{\Phi^+})$. Then*

$$\mathrm{Poin}\big(\mathrm{Hess}(\mathsf{N}, I), \sqrt{q}\big) = \mathrm{Dist}_{\mathcal{A}_I, R_0}(q).$$

Proof. Theorem 3.13 says that, for each $R \in C(\mathcal{A}_I)$ there is a unique element $w_R \in U(R)$ that belongs to $\mathrm{Sink}(I) = \{w \in W \mid w^{-1}(\Delta) \subset (-I) \cup \Phi^+\}$. Also, since $C_{w_R} \subset R$ and $F \subset R_0$, we have

$$\mathrm{Sep}(R, R_0) = \mathrm{Sep}(C_{w_R}, F) \cap \mathcal{A}_I = \{H_\alpha \mid \alpha \in N(w_R) \cap I\}.$$

Then we have

$$\mathrm{Dist}_{\mathcal{A}_I, R_0}(q) = \sum_{R \in C(\mathcal{A}_I)} q^{\mathrm{dist}(R, R_0)} = \sum_{w \in \mathrm{Sink}(I)} q^{|N(w) \cap I|} = \mathrm{Poin}\big(\mathrm{Hess}(\mathsf{N}, I), \sqrt{q}\big),$$

as desired. ☐

Example 3.16. Let $\Phi^+ = \Phi^+_{B_2}$. Then the sets $I_1 = \{\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_2\}$ and $I_2 = \{\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_2, \mathbf{e}_1\}$ are lower ideals of Φ^+. Theorem 3.15 and Figure 11 tell that the Poincaré polynomials of regular nilpotent Hessenberg varieties corresponding to these ideals are given by

$$\mathrm{Poin}\big(\mathrm{Hess}(\mathsf{N}, I_1), q\big) = 1 + 2q^2 + q^4$$

and

$$\mathrm{Poin}\big(\mathrm{Hess}(\mathsf{N}, I_2), q\big) = 1 + 2q^2 + 2q^4 + q^6.$$

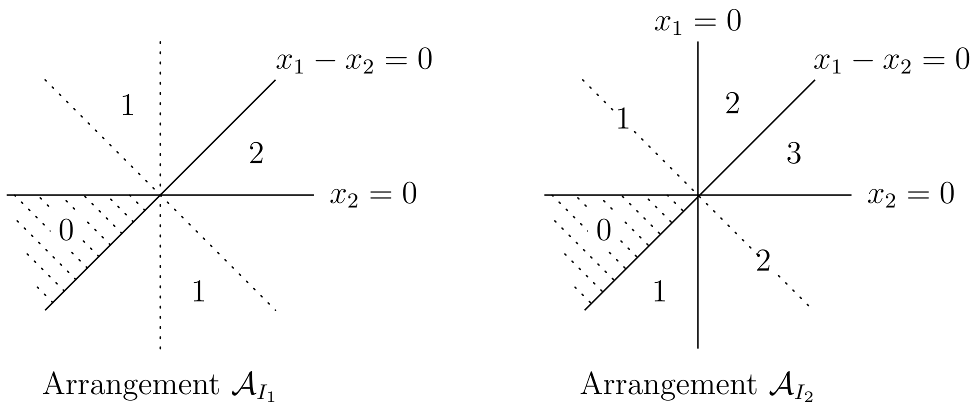


FIGURE 11. Arrangements $\mathcal{A}_{I_1}$ and $\mathcal{A}_{I_2}$. Shaded area is the chamber R_0 and numbers on chambers are distances from R_0.

3.6. **Weyl type subsets.** Theorem 3.15 is due to Sommers and Tymoczko [SoTy], but our formulation and proof are slightly different to the proof given by them. Indeed, given a lower ideal $I \subset \Phi^+$, Sommers and Tymoczko gave a nice criterion for a subset of $\mathcal{A}_I$ which equals to $\mathrm{Sep}(R, R_0)$ for some $R \in C(\mathcal{A}_I)$. We will not use this criterion in this chapter but introduce the result without proofs. Let $I \subset \Phi^+$ be a lower ideal of Φ^+. A subset $S \subset I$ is said to be of **Weyl-type** for I if it satisfies

(W1) $\alpha, \alpha' \in S$ and $\alpha + \alpha' \in I$ imply $\alpha + \alpha' \in S$, and
(W2) $\alpha, \alpha' \notin S$ and $\alpha + \alpha' \in I$ imply $\alpha + \alpha' \notin S$.

We write $\mathcal{W}^I$ for the set of all subsets of I of Weyl-type for I.

Theorem 3.17 (Sommers-Tymoczko)**.** *Let I be a lower ideal of Φ^+, $\mathcal{B} \subset \mathcal{A}_I$, and $R_0 \in C(\mathcal{A}_I)$ the chamber of $\mathcal{A}_I$ that contains the fundamental Weyl chamber $F \in C(\Phi^+)$. Then $\mathcal{B} = \mathrm{Sep}(R, R_0)$ for some $R \in C(\mathcal{A}_I)$ if and only if the set $\{\alpha \in I \mid H_\alpha \in \mathcal{B}\}$ is of Weyl type for I.*

The theorem clearly gives the following expression of Poincaré series of regular nilpotent Hessenberg varieties, which were more commonly used than the formulation in Theorem 3.15.

Corollary 3.18. *For a lower ideal $I \subset \Phi^+$, one has*

$$\mathrm{Poin}\big(\mathrm{Hess}(\mathsf{N}, I), \sqrt{q}\big) = \textstyle\sum_{S \in \mathcal{W}^I} q^{|S|}.$$

Remark 3.19. The coconvex property ($\star$) is not enough to obtain the conclusion of Theorem 3.17.

Note. We focus on Betti numbers of regular semisimple and regular nilpotent Hessenberg varieties, but a more general formula of Betti numbers for any regular Hessenberg variety is known. See [Pr13, Pr18].

The coconvex condition naturally appears when we consider the set $N(w)$. Indeed it is known that a subset $J \subset \Phi^+$ equals to $N(w)$ for some $w \in W$ if and only if J is convex (i.e., for all $\alpha, \beta \in J$ with $\alpha + \beta \in \Phi^+$ one has $\alpha + \beta \in J$) and coconvex. The proof of Theorem 3.12 given in this section is based on the idea appeared in the discussion with Abe, Horiguchi, Masuda and Sato. Some more properties of arrangements associated with coconvex subsets of Φ^+ were studied in [Sl16, TrTs].

4. Free arrangements

In the previous section, we see that Poincaré polynomials of regular semisimple and regular nilpotent Hessenberg varieties can be computed using distances of chambers of corresponding ideal arrangements. The next goal of this chapter is to explain that, for regular nilpotent Hessenberg varieties, one can determine not only their Poincaré polynomials but also the ring structure of their cohomology rings from ideal arrangements. This comes from the theory of free hyperplane arrangements, so we quickly review the freeness of arrangements in this section. We will not visit a deep algebraic aspect of this theory. Instead, we introduce basic results without proofs and explain how the results are related to combinatorics of arrangements and invariant theory. For readers who want to learn more on freeness of arrangements, see the book of Orlik and Terao [OT].

4.1. **Definition and basic results.** We recall our setting. Let $V \cong \mathbb{R}^n$ be an n-dimensional Euclidian space with an inner product $(-,-)$, $\mathbf{e}_1, \dots, \mathbf{e}_n$ an orthonormal basis of V, and $x_1, \dots, x_n$ the corresponding dual basis in the dual space V^*. We write $\mathcal{R} = \mathrm{sym}(V^*) = \mathbb{R}[x_1, \dots, x_n]$ for the symmetric algebra of V^* with each $\deg(x_i) = 1$. Let $\mathrm{Der}(\mathcal{R})$ be the module of derivations of $\mathcal{R}$. It is known that $\mathrm{Der}(\mathcal{R})$ has the decomposition

$$\mathrm{Der}(\mathcal{R}) = \mathcal{R}\left(\frac{\partial}{\partial x_1}\right) \oplus \mathcal{R}\left(\frac{\partial}{\partial x_2}\right) \oplus \cdots \oplus \mathcal{R}\left(\frac{\partial}{\partial x_n}\right)$$

as $\mathcal{R}$-modules. Thus $\mathrm{Der}(\mathcal{R})$ is, as an $\mathcal{R}$-module, the free $\mathcal{R}$-module of rank n with a free $\mathcal{R}$-basis $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\}$, and acts on $\mathcal{R}$ as follows: For $\theta = \sum_{k=1}^n f_k \frac{\partial}{\partial x_k} \in \mathrm{Der}(\mathcal{R})$, one has $\theta \cdot g = \sum_{k=1}^n f_k (\frac{\partial}{\partial x_k} g)$ for $g \in \mathcal{R}$. We define the grading of $\mathrm{Der}(\mathcal{R})$ by setting $\deg(\frac{\partial}{\partial x_k}) = 0$ for each k. Thus, $f = \sum_{k=1}^n f_k \frac{\partial}{\partial x_k} \in \mathrm{Der}(\mathcal{R})$ is a homogeneous element of degree d if $\deg f_1 = \cdots = \deg f_n = d$. To simplify notation, we will write $\partial_k = \frac{\partial}{\partial x_k}$ for $k = 1, 2, \dots, n$.

The **logarithmic derivation module** $\mathcal{D}(\mathcal{A})$ (also called the **logarithmic vector field**) of $\mathcal{A}$ is the submodule of $\mathrm{Der}(\mathcal{R})$ defined by

$$\mathcal{D}(\mathcal{A}) = \{\theta \in \mathrm{Der}(\mathcal{R}) \mid \theta \cdot \ell_H \in \ell_H \mathcal{R} \quad \text{for all } H \in \mathcal{A}\}$$

where ℓ_H is a (fixed) defining linear form of a hyperplane H and $\ell_H\mathcal{R}$ is the ideal of $\mathcal{R}$ generated by ℓ_H. This module $\mathcal{D}(\mathcal{A})$ is nothing but the polynomial vector fields tangent to hyperplanes in $\mathcal{A}$, which is explained in the introduction. Indeed, for any $\theta = f_1\partial_1 + \cdots + f_n\partial_n \in \mathrm{Der}(\mathcal{R})$ and a hyperplane H in $\mathbb{R}^n$, it is easy to check

$$\theta \cdot \ell_H \in \ell_H\mathcal{R} \;\Leftrightarrow\; \big(f_1(\boldsymbol{a}), \dots, f_n(\boldsymbol{a})\big) \in H \text{ for all } \boldsymbol{a} \in H.$$

Example 4.1. Let $\mathcal{A} = \{H_{ij} \mid 1 \le i < j \le n\}$ be the braid arrangement in $\mathbb{R}^n$. We set $\ell_{H_{ij}} = x_i - x_j$. Consider the elements $\theta_1, \dots, \theta_n \in \mathrm{Der}(\mathcal{R})$ defined by

$$\theta_k = \sum_{i=1}^{n} x_i^{k-1}\partial_i \quad (k = 1, 2, \dots, n). \tag{4.1}$$

These elements belong to $\mathcal{D}(\mathcal{A})$. Indeed, one can easily see that

$$\theta_k \cdot (x_i - x_j) = x_i^{k-1} - x_j^{k-1} \in (x_i - x_j)\mathcal{R}$$

for all $1 \le i < j \le n$ and $1 \le k \le n$. We will soon see that the above $\theta_1, \dots, \theta_n$ actually generate $\mathcal{D}(\mathcal{A})$.

It is easy to see that $\mathcal{D}(\mathcal{A})$ is a graded $\mathcal{R}$-module (recall that we always assume that $\mathcal{A}$ is linear). Also, the definition clearly tells

$$Q(\mathcal{A}) \cdot \mathrm{Der}(\mathcal{R}) \subset \mathcal{D}(\mathcal{A}) \subset \mathrm{Der}(\mathcal{R}) \tag{4.2}$$

where $Q(\mathcal{A}) = \prod_{H \in \mathcal{A}} \ell_H$ is a defining polynomial of $\mathcal{A}$.

We say that $\mathcal{A}$ is a **free arrangement** if $\mathcal{D}(\mathcal{A})$ is a free $\mathcal{R}$-module. Recall that the rank of a free $\mathcal{R}$-module M equals to the $\mathcal{Q}$-dimension of $M \otimes_{\mathcal{R}} \mathcal{Q}$, where $\mathcal{Q}$ is the field of fractions of $\mathcal{R}$. Suppose that $\mathcal{A}$ is a free arrangement. Then (4.2) tells that $\mathcal{D}(\mathcal{A})$ must have rank n (as $Q(\mathcal{A}) \cdot \mathrm{Der}(\mathcal{R}) \otimes_{\mathcal{R}} \mathcal{Q} = \mathrm{Der}(\mathcal{R}) \otimes_{\mathcal{R}} \mathcal{Q} \cong \mathcal{Q}^n$), so there are homogeneous elements $\theta_1, \dots, \theta_n \in \mathcal{D}(\mathcal{A})$ that generate $\mathcal{D}(\mathcal{A})$. Let $d_k = \deg \theta_k$ for all k. We call the sequence $(d_1, \dots, d_n)$ the **exponents** of $\mathcal{A}$.

How can we prove the freeness of the $\mathcal{R}$-module $\mathcal{D}(\mathcal{A})$? The next criterion of the freeness, known as Saito's criterion, gives a simple methods to prove the freeness of arrangements when we know a candidate of a free $\mathcal{R}$-basis.

Theorem 4.2 (Saito's criterion)**.** *Let $\mathcal{A}$ be an arrangement in V and $\theta_1, \dots, \theta_n \in \mathcal{D}(\mathcal{A})$. Then $\mathcal{D}(\mathcal{A})$ is a free $\mathcal{R}$-module with a free $\mathcal{R}$-basis $\theta_1, \dots, \theta_n$ if and only if $\theta_1, \dots, \theta_n$ is $\mathcal{R}$-independent and $\sum_{k=1}^{n} \deg(\theta_k) = |\mathcal{A}|$.*

See [OT, Proposition 4.12 and Theorem 4.19] for the proof of the above criterion. To check the $\mathcal{R}$-independentness, it is convenient to consider determinants. For $\theta_1, \dots, \theta_n \in \mathrm{Det}(\mathcal{R})$ with $\theta_i = \sum_{j=1}^{n} f_{ij}\partial_j$, we define $\det(\theta_1, \dots, \theta_n)$ to be the determinant of the $n \times n$ matrix $(f_{ij})_{1 \le i,j \le n}$. Then $\theta_1, \dots, \theta_n$ is $\mathcal{R}$-independent if and only if $\det(\theta_1, \dots, \theta_n)$ is non-zero[7].

Example 4.3. Consider again the braid arrangement $\mathcal{A} = \{H_{ij} \mid 1 \le i < j \le n\}$ in $\mathbb{R}^n$. Then the elements $\theta_1, \dots, \theta_n \in \mathcal{D}(\mathcal{A})$ given in (4.1) satisfy

[7]It is known that if $\theta_1, \dots, \theta_n \in \mathcal{D}(\mathcal{A})$, then $\det(\theta_1, \dots, \theta_n)$ equals to $Q(\mathcal{A}) \cdot f$ for some $f \in \mathcal{R}$. See [OT, Proposition 4.12].

- The determinant $\det(\theta_1, \dots, \theta_n)$ is the Vandermond determinant. In particular, it is non-zero and therefore $\theta_1, \dots, \theta_n$ are $\mathcal{R}$-independent.
- $\sum_{k=1}^{n} \deg \theta_k = \sum_{k=1}^{n}(k-1) = n(n-1)/2 = |\mathcal{A}|$.

Then Saito's criterion tells that $\mathcal{D}(\mathcal{A})$ is a free $\mathcal{R}$-module with an $\mathcal{R}$-basis $\theta_1, \dots, \theta_n$. In particular $\mathcal{A}$ is a free arrangement with exponents $(0, 1, \dots, n-1)$.

Example 4.4. We give an example of an arrangement which is not free. Consider the arrangement $\mathcal{A} = \{H_{12}, H_{23}, H_{34}, H_{14}\}$ in $\mathbb{R}^4$ (graphic arrangement of the 4-cycle). Then $\mathcal{D}(\mathcal{A})$ is generated by the following 5 elements

$$\begin{aligned}
\theta_1 &= \partial_1 + \cdots + \partial_4, \\
\theta_2 &= x_1\partial_1 + \cdots + x_4\partial_4, \\
\theta_3 &= (x_1 - x_2)(x_2 - x_3)\partial_2, \\
\theta_4 &= (x_2 - x_3)(x_3 - x_4)\partial_3, \\
\theta_5 &= (x_1 - x_2)(x_3 - x_4)\partial_2 + (x_1 - x_3)(x_3 - x_4)\partial_3.
\end{aligned}$$

These elements are not $\mathcal{R}$-independent. Indeed, we have a non-trivial relation

$$(x_3 - x_4)\theta_3 + (x_1 - x_3)\theta_4 - (x_2 - x_3)\theta_5 = 0.$$

so the arrangement $\mathcal{A}$ is not free.

The most important result on free arrangements would be the following **Terao's factorization theorem**, which gives a beautiful formula of characteristic polynomials of free arrangements.

Theorem 4.5 (Terao's factorization)**.** *If $\mathcal{A}$ is a free arrangement with exponents $(d_1, \dots, d_n)$, then*

$$\chi_{\mathcal{A}}(q) = \textstyle\prod_{k=1}^{n}(q - d_k).$$

The proof of Terao's factorization theorem is not easy, so we omit it in this chapter and refer the readers to [OT, §4.6].

Example 4.6. We already see that the braid arrangement in $\mathbb{R}^n$ is a free arrangement with exponents $(0, 1, \dots, n-1)$. Terao's factorization guarantees that its characteristic polynomial is given by

$$q(q-1)(q-2)\cdots(q-n+1).$$

One can see that this coincide with the computation of the characteristic polynomial of the braid arrangement given in §2.5.

4.2. **Freeness of Weyl arrangements.** Logarithmic derivation modules of Weyl arrangements are closely related to classical invariant theory. Here we will explain how the freeness of Weyl arrangements follows from the invariant theory.

Let Φ be an irreducible root system in $V \cong \mathbb{R}^n$. Recall that the Weyl group W is the subgroup of $\mathrm{GL}(V)$ generated by all reflections s_α with $\alpha \in \Phi$. We consider the action of $\mathrm{GL}(V)$ (and therefore of W) on $\mathcal{R}$ defined by

$$g \cdot f(\boldsymbol{x}) = f(g^{-1}(\boldsymbol{x}))$$

for any $g \in \mathrm{GL}(V)$, $f \in \mathcal{R}$ and $\boldsymbol{x} \in V$, and consider the ring of invariants

$$\mathcal{R}^W = \{f \in \mathcal{R} \mid w \cdot f = f \quad \text{for any } w \in W\}.$$

By Chevalley's theorem (see [Hum2, §3.5]), there are homogeneous polynomials $P_1, \dots, P_n \in \mathcal{R}^W$ such that

$$\mathcal{R}^W = \mathbb{R}[P_1, \dots, P_n].$$

The polynomials $P_1, \dots, P_n$ are called **basic invariants** for W. Below, we list some standard properties of basic invariants (see [Hum2, §3] or [OT, Chapter 6]).

(I1) Let $m_k = \deg P_k - 1$ for $k = 1, 2, \dots, n$. Then $\sum_{k=1}^n m_k = |\Phi^+|$.
(I2) The Jacobian $\det(\frac{\partial}{\partial x_j} P_i)_{1 \le i,j \le n}$ is a non-zero constant multiple of $\prod_{\alpha \in \Phi^+} \ell_\alpha$.
(I3) $\dim_{\mathbb{R}}(\mathcal{R}/(P_1, \dots, P_n)) = |W|$.

The sequence $(m_1, \dots, m_n)$ is called the **exponents** of the Weyl group W. We will later see that this coincide with the exponents of the Weyl arrangement $\mathcal{A}_{\Phi^+}$.

Example 4.7. Below are basic invariants for the root systems in Table 1. See [Hum2, §3.12].

- For type A_{n-1}, one possible choice of basic invariants are[8]

$$P_k = \sum_{k=1}^n x_i^k \quad (k = 2, 3, \dots, n).$$

- For type B_n and C_n (they have the same Weyl group), one possible choice of basic invariants are

$$P_k = \sum_{i=1}^n x_i^{2k} \quad (k = 1, 2, \dots, n).$$

- For type D_n, one possible choice of basic invariants are

$$P_k = \sum_{k=1}^n x_i^{2k} \quad (k = 1, 2, \dots, n-1)$$

and

$$P_n = x_1 x_2 \cdots x_n.$$

We extend the action of $\mathrm{GL}(V)$ (and therefore of W) on $\mathcal{R}$ to the module of derivations $\mathrm{Der}(\mathcal{R})$ by

$$g \cdot \left(\sum_{k=1}^n f_k \frac{\partial}{\partial x_k} \right) = \sum_{k=1}^n (g \cdot f_k) \frac{\partial}{\partial (g \cdot x_k)}$$

where we write $\frac{\partial}{\partial \ell} = a_1 \frac{\partial}{\partial x_1} + \cdots + a_n \frac{\partial}{\partial x_n}$ for $\ell = a_1 x_1 + \cdots + a_n x_n$[9]. Consider the invariant space

$$\mathrm{Der}(\mathcal{R})^W = \{\theta \in \mathrm{Der}(\mathcal{R}) \mid w \cdot \theta = w \text{ for any } w \in W\}$$

[8]For type A_{n-1}, we consider that $\mathcal{R} = \mathrm{sym}(V^*) = \mathbb{R}[x_1, \dots, x_n]/(x_1 + \cdots + x_n)$ since V is the hyperplane in $\mathbb{R}^n$ defined by $x_1 + \cdots + x_n = 0$ although $x_1, \dots, x_n$ are not orthonormal basis for V^* in this case.

[9]$\frac{\partial}{\partial \ell}$ may be regarded as the derivation along the vector $\alpha_\ell = a_1 \mathbf{e}_1 + \cdots + a_n \mathbf{e}_n$, that is, $\frac{\partial}{\partial \ell}$ is the derivation defined by $\frac{\partial}{\partial \ell} f(\boldsymbol{x}) = \lim_{h \to 0} \frac{f(\boldsymbol{x} + h\alpha_\ell) - f(\boldsymbol{x})}{h}$

by the above action. We note that $\mathrm{Der}(\mathcal{R})^W$ is not an $\mathcal{R}$-module but is an $\mathcal{R}^W$-module.

There is a nice relation among $\mathcal{R}^W$, $\mathrm{Der}(\mathcal{R})^W$ and $\mathcal{D}(\mathcal{A}_{\Phi^+})$. For $f \in \mathcal{R}$, let

$$df = \sum_{k=1}^{n} \left(\frac{\partial}{\partial x_k} f \right) \partial_k \in \mathrm{Der}(\mathcal{R})$$

be the total derivation of f. The total derivation does not depend on a choice of an orthonormal basis $x_1, \ldots, x_n$ of V^* in the following sense: Given any orthogonal transformation $g \in \mathrm{GL}(V)$ with matrix presentation $(g \cdot x_1, \ldots, g \cdot x_n) = (x_1, \ldots, x_n)P$, where P is an orthogonal matrix, we have

$$\sum_{k=1}^{n} \left(\left(\frac{\partial}{\partial (g \cdot x_k)} \cdot f \right) \frac{\partial}{\partial (g \cdot x_k)} \right) = (\partial_1 f, \ldots, \partial_n f) P \, {}^tP \, {}^t(\partial_1, \ldots, \partial_n) = \sum_{k=1}^{n} \left(\frac{\partial}{\partial x_k} f \right) \partial_k.$$

This implies the following property.

Lemma 4.8. *If $f \in \mathcal{R}^W$ then $df \in \mathrm{Der}(\mathcal{R})^W$.*

Proof. Let $g \in W$. The statement follows from the following computation

$$g \cdot (df) = \sum_{k=1}^{n} \left(\frac{\partial}{\partial (g \cdot x_k)} (g \cdot f) \right) \left(\frac{\partial}{\partial (g \cdot x_k)} \right) = d(g \cdot f) = df,$$

where we use the fact that the total derivation does not depend on a choice of an orthonormal basis for the second equality. □

We now consider the logarithmic derivation module $\mathcal{D}(\mathcal{A}_{\Phi^+})$. It is easy to see

$$\mathrm{Der}(R)^W \subset \mathcal{D}(\mathcal{A}_{\Phi^+}). \tag{4.3}$$

Indeed, for any $\theta \in \mathrm{Der}(\mathcal{R})^W$ and $\alpha \in \Phi$, we have

$$s_\alpha \cdot (\theta \cdot \ell_\alpha) = \theta \cdot (s_\alpha \cdot \ell_\alpha) = -\theta \cdot \ell_\alpha,$$

which means that $\theta \cdot \ell_\alpha(\boldsymbol{a}) = 0$ for any $\alpha \in \Phi$ and $\boldsymbol{a} \in H_\alpha$, proving $\theta \in \mathcal{D}(\mathcal{A}_{\Phi^+})$. (Recall for $\mathbf{0} \neq \alpha = \sum_{k=1}^n a_k \mathbf{e}_k \in V$, we write $\ell_\alpha = \sum_{k=1}^n a_k x_k$.) Let $P_1, \ldots, P_n$ be basic invariants for W. By Lemma 4.8 and (4.3), $dP_1, \ldots, dP_n$ are elements of $\mathcal{D}(\mathcal{A}_{\Phi^+})$. These elements actually give an $\mathcal{R}$-basis for $\mathcal{D}(\mathcal{A}_{\Phi^+})$.

Theorem 4.9 (K. Saito [Sa75, Sa80])**.** *The module $\mathcal{D}(\mathcal{A}_{\Phi^+})$ is a free $\mathcal{R}$-module with a free $\mathcal{R}$-basis $dP_1, \ldots, dP_n$.*

Proof. By the property (I1) of basic invariants, we have

$$\textstyle\sum_{i=1}^n \deg(dP_i) = \sum_{i=1}^n (\deg P_i - 1) = |\Phi^+| = |\mathcal{A}_{\Phi^+}|.$$

Also, by the property (I2), the determinant of $(\frac{\partial}{\partial x_i} P_j)_{1 \leq i,j \leq n}$ is non-zero, which implies that $dP_1, \ldots, dP_n$ are $\mathcal{R}$-independent. Now Saito's criterion guarantees the desired assertion. □

Remark 4.10. It is also known that the invariant space $\mathrm{Der}(\mathcal{R})^W$ is a free $\mathcal{R}^W$-module with a free $\mathcal{R}^W$-basis $dP_1, \ldots, dP_n$. See [OT, §6.3].

Remark 4.11. Since $\deg dP_k = m_k$, the exponents of the arrangement $\mathcal{A}_{\Phi^+}$ coincides with the exponents of the Weyl group W.

Example 4.12. Using basic invariants $P_1, \dots, P_n$ given in Example 4.7, we can see that the following elements give a basis of $\mathcal{D}(\mathcal{A}_{\Phi^+})$ for Lie types A_{n-1}, B_n, C_n and D_n.

- For type A_{n-1}[10],
$$\theta_k = \sum_{i=1}^{n} x_i^{k-1}\partial_i \quad (k = 2, 3, \dots, n).$$
- For type B_n and C_n,
$$\theta_k = \sum_{i=1}^{n} x_i^{2k-1}\partial_i \quad (k = 1, 2, \dots, n).$$
- For type D_n,
$$\theta_k = \sum_{i=1}^{n} x_i^{2k-1}\partial_i \quad (k = 1, 2, \dots, n-1)$$
and
$$\theta_n = x_1 \cdots x_n \left(\sum_{i=1}^{n} \frac{1}{x_i}\partial_i \right).$$

4.3. **Freeness of ideal arrangements.** To explain a relation between the cohomology $H^*(\mathrm{Hess}(\mathsf{N}, I))$ and the ideal arrangement $\mathcal{A}_I$, we need the freeness of $\mathcal{A}_I$. The freeness of ideal arrangements was conjectured by Sommers and Tymoczko [SoTy], who also proved the conjecture except for types F_4, E_6, E_7, E_8. The conjecture was later proved by Abe, Barakat, Cuntz, Hoge and Terao [ABCHT] using a type free argument. We will not go into the proof of this result and refer the readers to [ABCHT] (we give a proof for classical types A_{n-1}, B_n, C_n and D_n later in section 8 by constructing a basis of $\mathcal{D}(\mathcal{A}_I)$ explicitly). Instead we quickly introduce the statement and explain what is the exponents of $\mathcal{A}_I$.

For $\alpha \in \Phi$ with $\alpha = a_1\alpha_1 + \cdots + a_n\alpha_n$, where $\alpha_1, \dots, \alpha_n$ are simple roots, the **height** of α is
$$\mathrm{ht}(\alpha) = a_1 + \cdots + a_n.$$
Let $I \subset \Phi^+$ be a lower ideal of a root system Φ^+. The **height distribution** $\mathrm{ht}(I)$ of I is the sequence
$$\mathrm{ht}(I) = (h_1^I, h_2^I, h_3^I, \cdots)$$
defined by $h_k^I = |\{\alpha \in I \mid \mathrm{ht}(\alpha) = k\}|$. We note that $n \ge h_1^I \ge h_2^I \ge h_3^I \ge \cdots$ and we have $n = h_1^I$ if $I \supset \Delta$. The **ideal exponents** of I is the sequence $d(I) = (d_1^I, \dots, d_n^I)$ defined by
$$d_k^I = |\{j \in \mathbb{N} \mid h_j^I \ge k\}|.$$
In other words, $d(I)$ is the dual partition of $\mathrm{ht}(I)$. We note that d_k^I could be zero.

[10]Recall again that we take $\mathcal{R} = \mathbb{R}[x_1, \dots, x_n]/(x_1 + \cdots + x_n)$ for type A_{n-1}.

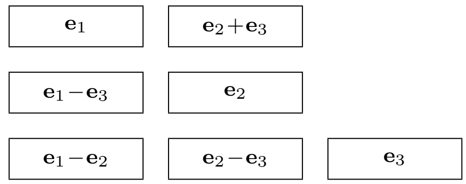


FIGURE 12. Visualization of the ideal I in Example 4.14. We may consider that $\mathrm{ht}(I)$ counts the number of elements in each low and $d(I)$ counts the number of each column.

Theorem 4.13 (Sommers-Tymoczko, Abe-Barakat-Cuntz-Hoge-Terao)**.** *For any lower ideal $I \subset \Phi^+$, the arrangement $\mathcal{A}_I$ is free with exponents $d(I)$.*

Example 4.14. Let

$$I = \{\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_1 - \mathbf{e}_3, \mathbf{e}_1, \mathbf{e}_2 - \mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_3\} \subset \Phi^+_{B_3}.$$

Then I is an ideal of $\Phi^+_{B_3}$, and it has 3 elements of height 1, 2 elements of height 2, 2 elements of height 3, so $\mathrm{ht}(I) = (3, 2, 2)$ and $d(I) = (3, 3, 1)$. See Figure 12 for a visualization of these sequences.

Note. In general, it is not easy to check if a given arrangement $\mathcal{A}$ is free or not. For graphic arrangements (or subarrangements of type A Weyl arrangements), their freeness is combinatorially characterized. Indeed, it is known that a graphic arrangement $\mathcal{A}_G$ is free if and only if the underlying graph G is a chordal graph (see [ER, Theorem 3.3]). However, it is still an open problem to get a combinatorial characterization of the freeness for subarrangements of type B Weyl arrangements. See [ER, STT, ToTs] and references therein for background and recent results on this topic. A much bigger open problem on this subject is Terao's conjecture asking if the freeness of the arrangement $\mathcal{A}$ depends only on the poset $\mathcal{L}_{\mathcal{A}}$. See [Yo, §1.7] for more information on the conjecture.

The study of ideal arrangements seem to start from the work in [SoTy]. See [AR, CRS, Röh] for more studies on ideal arrangements.

5. Coinvariant algebra and Solomon–Terao algebra

Let V, Φ, W and $\mathcal{R}$ be as in the previous section. Let $(\mathcal{R}^W_+)$ be the ideal of $\mathcal{R}$ generated by all polynomials in $\mathcal{R}^W$ of positive degrees. Since the basic invariants $P_1, \dots, P_n$ generates $\mathcal{R}^W$, we clearly have $(\mathcal{R}^W_+) = (P_1, \dots, P_n)$. The **coinvariant algebra** for W is the quotient ring

$$\mathcal{R}/(\mathcal{R}^W_+) = \mathcal{R}/(P_1, \dots, P_n).$$

Coinvariant algebras are fundamental in Schubert calculus since it is isomorphic to the cohomology rings of flag varieties.

We explain that coinvariant algebras can be constructed from logarithmic derivation modules of Weyl arrangements. Consider the $\mathcal{R}$-homomorphism $\eta : \mathrm{Der}(\mathcal{R}) \to \mathcal{R}$ defined by

$$\eta(f_1\partial_1 + \cdots + f_n\partial_n) = f_1x_1 + \cdots + f_nx_n.$$

The **Solomon–Terao ideal** $\mathfrak{a}(\mathcal{A})$ of an arrangement $\mathcal{A}$ in V is the image of $\mathcal{D}(\mathcal{A})$ by η, that is,

$$\mathfrak{a}(\mathcal{A}) := \{\eta(\theta) \mid \theta \in \mathcal{D}(\mathcal{A})\}.$$

The quotient ring $\mathcal{R}/\mathfrak{a}(\mathcal{A})$ is called the **Solomon–Terao algebra** of $\mathcal{A}$.

Proposition 5.1. *The Solomon–Terao algebra $\mathcal{R}/\mathfrak{a}(\mathcal{A}_{\Phi^+})$ of the Weyl arrangement $\mathcal{A}_{\Phi^+}$ equals to the coinvariant algebra $\mathcal{R}/(\mathcal{R}_+^W)$.*

Proof. Recall that $\mathcal{D}(\mathcal{A}_{\Phi^+})$ is generated by $dP_1, \dots, dP_n$, where $P_1, \dots, P_n$ are basic invariants of the Weyl group W. Since

$$\eta(df) = (\deg f) \times f$$

for any homogeneous polynomial $f \in \mathcal{R}$, the ideal $\mathfrak{a}(\mathcal{A}_{\Phi^+})$ is nothing but the ideal generated by $P_1, \dots, P_n$, and the Solomon–Terao algebra of $\mathcal{A}_{\Phi^+}$ is the coinvariant algebra $\mathcal{R}/(P_1, \dots, P_n)$. □

By the above proposition, Solomon–Terao algebra may be considered as an arrangement theoretic generalization of coinvariant algebras. Below we study some properties of Solomon–Terao algebras.

5.1. **Basic properties of Solomon–Terao algebras.** We need some known facts on regular sequences for $\mathcal{R}$. Let $\mathfrak{a} \subset \mathcal{R}$ be a homogeneous ideal. A graded $\mathbb{R}$-algebra $\mathcal{R}/\mathfrak{a}$ is **Artinian** if $\dim_{\mathbb{R}} \mathcal{R}/\mathfrak{a} < \infty$. A sequence of homogeneous polynomial $f_1, \dots, f_m \in \mathcal{R}$ of positive degrees is said to be a **regular sequence** for $\mathcal{R}$ if f_k is a non-zero divisor of $\mathcal{R}/(f_1, \dots, f_{k-1})$ for $k = 1, 2, \dots, m$. If $f_1, \dots, f_m \in \mathcal{R}$ is a regular sequence, then the quotient ring $\mathcal{R}/(f_1, \dots, f_m)$ is called a **complete intersection**. Below are some properties of regular sequences which we will need (see Appendix for proofs).

Lemma 5.2. *Let $f_1, \dots, f_n \in \mathcal{R}$ be homogeneous polynomials of positive degrees. If $\mathcal{R}/(f_1, \dots, f_n)$ is Artinian then $f_1, \dots, f_n$ is a regular sequence for $\mathcal{R}$.*

The **Hilbert series** of a graded $\mathcal{R}$-module M is the formal power series

$$\mathrm{Hilb}(M, q) := \textstyle\sum_{k \geq 0} (\dim_{\mathbb{R}} M_k)\, q^k,$$

where M_k is the homogeneous component of M of degree k.

Lemma 5.3. *Let $f_1, \dots, f_n \in \mathcal{R}$ be a regular sequence, $d_k = \deg f_k$ for $k = 1, 2, \dots, n$, and $s = \sum_{k=1}^n (d_k - 1)$. Then*

(1) $\mathrm{Hilb}(\mathcal{R}/(f_1, \dots, f_n), q) = \prod_{k=1}^n (1 + q + \cdots + q^{d_k - 1})$.
(2) *Let Δ be the determinant of the $n \times n$ matrix $(\frac{\partial}{\partial x_j} f_i)_{1 \leq i,j \leq n}$, then Δ is an $\mathbb{R}$-basis for $(\mathcal{R}/(f_1, \dots, f_n))_s$.*

We note that (1) tells $(\mathcal{R}/(f_1, \dots, f_n))_s \cong \mathbb{R}$.

Theorem 5.4. *Let $\mathcal{A}$ be an arrangement in V. Then*

(1) *$\mathcal{R}/\mathfrak{a}(\mathcal{A})$ is Artinian.*
(2) *A defining polynomial $\prod_{H \in \mathcal{A}} \ell_H$ of $\mathcal{A}$ is non-zero in $\mathcal{R}/\mathfrak{a}(\mathcal{A})$.*

(3) *If* $\mathcal{A}$ *is a free arrangement with exponents* $(d_1, \dots, d_n)$ *then* $\mathcal{R}/\mathfrak{a}(\mathcal{A})$ *is a complete intersection with*

$$\mathrm{Hilb}\big(\mathcal{R}/\mathfrak{a}(\mathcal{A}), q\big) = \textstyle\prod_{k=1}^{n}(1 + q + \cdots + q^{d_k}).$$

(4) *If* $\mathcal{A}$ *is a free arrangement, then* $\dim_{\mathbb{R}}(\mathcal{R}/\mathfrak{a}(\mathcal{A}))$ *equals to the number of chambers of* $\mathcal{A}$*.*

Proof. While the theorem holds for arbitrary arrangement, to simplify proofs, we only prove it for the case when $\mathcal{A}$ is a subarrangement of the Weyl arrangement $\mathcal{A}_{\Phi^+}$, that is, the case when $\mathcal{A} = \mathcal{A}_J$ for some $J \subset \Phi^+$. See [AMMN] for proofs of a general case.

(1) Since $\mathcal{A}_J \subset \mathcal{A}_{\Phi^+}$ implies $\mathcal{D}(\mathcal{A}_J) \supset \mathcal{D}(\mathcal{A}_{\Phi^+})$ and $\mathfrak{a}(\mathcal{A}_J) \supset \mathfrak{a}(\mathcal{A}_{\Phi^+})$, we have

$$\dim_{\mathbb{R}}(\mathcal{R}/\mathfrak{a}(\mathcal{A}_J)) \le \dim_{\mathbb{R}}(\mathcal{R}/\mathfrak{a}(\mathcal{A}_{\Phi^+})) = \dim_{\mathbb{R}}(\mathcal{R}/(\mathcal{R}_+^W)) = |W| < \infty,$$

where we use (I3) for the last equality.

(2) We note that $\prod_{\alpha \in J} \ell_\alpha$ is a defining polynomial of $\mathcal{A}_J$. Let $P_1, \dots, P_n$ be basic invariants for W. Recall that $\det(\frac{\partial}{\partial x_j} P_i)$ is a constant multiple of $\prod_{\alpha \in \Phi^+} \ell_\alpha$. Then, since $\mathcal{R}/(P_1, \dots, P_n)$ is Artinian, by Lemmas 5.2 and 5.3, the sequence $P_1, \dots, P_n$ is a regular sequence for $\mathcal{R}$ and $\prod_{\alpha \in \Phi^+} \ell_\alpha$ is non-zero in $\mathcal{R}/(P_1, \dots, P_n)$.

Let $\beta_J = \prod_{\alpha \in \Phi^+ \setminus J} \ell_\alpha$. The definition of logarithmic derivation modules tells

$$\begin{aligned}\beta_J \cdot \mathcal{D}(\mathcal{A}_J) &= \{\beta_J \cdot \theta \in \mathrm{Der}(\mathcal{R}) \mid \theta \cdot \ell_\alpha \in \ell_\alpha \mathcal{R} \text{ for all } \alpha \in J\} \\ &\subset \{\theta \in \mathrm{Der}(\mathcal{R}) \mid \theta \cdot \ell_\alpha \in \ell_\alpha \mathcal{R} \text{ for all } \alpha \in \Phi^+\} \\ &= \mathcal{D}(\mathcal{A}_{\Phi^+}),\end{aligned}$$

where the second inclusion follows from the fact that $\beta_J \in \ell_\alpha \mathcal{R}$ for any $\alpha \in \Phi^+ \setminus J$. This inclusion induces the inclusion

$$\beta_J \mathfrak{a}(\mathcal{A}_J) \subset \mathfrak{a}(\mathcal{A}_{\Phi^+}) = (P_1, \dots, P_n).$$

Then we have $\prod_{\alpha \in J} \ell_\alpha \notin \mathfrak{a}(\mathcal{A}_J)$ since $\beta_J \cdot \prod_{\alpha \in J} \ell_\alpha = \prod_{\alpha \in \Phi^+} \ell_\alpha \notin (P_1, \dots, P_n)$.

(3) If $\mathcal{A}_J$ is free with exponents $(d_1, \dots, d_n)$, then $\mathcal{D}(\mathcal{A}_J)$ is generated by elements $\theta_1, \dots, \theta_n$ with $\deg \theta_k = d_k$ for $k = 1, 2, \dots, n$. Then since $\mathcal{R}/\mathfrak{a}(\mathcal{A}_J) = \mathcal{R}/(\eta(\theta_1), \dots, \eta(\theta_n))$ is Artinian by (1), the sequence $\eta(\theta_1), \dots, \eta(\theta_n)$ is a regular sequence by Lemma 5.2, so $\mathcal{R}/\mathfrak{a}(\mathcal{A}_J)$ is a complete intersection. The formula of Hilbert series follows from Lemma 5.3(1).

(4) This last statement follows from (3) since it implies

$$\dim_{\mathbb{R}}\big(\mathcal{R}/\mathfrak{a}(\mathcal{A}_J)\big) = \mathrm{Hilb}\big(\mathcal{R}/\mathfrak{a}(\mathcal{A}_J), 1\big) = \textstyle\prod_{k=1}^{n}(1 + d_k) = |\chi_{\mathcal{A}_J}(-1)| = |C(\mathcal{A}_J)|,$$

where we use Terao's factorization and Zaslabsky's theorem for the last two equalities. □

Remark 5.5. The converse of Theorem 5.4(3) also holds. If $\mathcal{R}/\mathfrak{a}(\mathcal{A})$ is a complete intersection, then $\mathcal{A}$ must be free. See [AMMN, ES].

5.2. **Solomon–Terao algebra of subarrangements of Weyl arrangements.** In this subsection, we show that the Solomon–Terao algebra $\mathcal{R}/\mathfrak{a}(\mathcal{A})$ has a simple expression in terms of ideal quotients when $\mathcal{A}$ is a free subarrangement of the Weyl arrangement $\mathcal{A}_{\Phi^+}$.

For an ideal $\mathfrak{a} \subset \mathcal{R}$ and $f \in \mathcal{R}$, the **ideal quotient** $(I : f)$ is the ideal of $\mathcal{R}$ defined by

$$(\mathfrak{a} : f) = \{g \in \mathcal{R} \mid fg \in \mathfrak{a}\}.$$

For $J \subset \Phi^+$, we define $\beta_J = \prod_{\alpha \in \Phi^+ \setminus J} \ell_\alpha$. Our goal is to prove the following presentation of Solomon–Terao algebras for free subarrangements of Weyl arrangements.

Theorem 5.6. *Let $J \subset \Phi^+$. If $\mathcal{A}_J$ is a free arrangement, then*

$$\mathcal{R}/\mathfrak{a}(\mathcal{A}_J) = \mathcal{R}/\big((\mathcal{R}^W_+) : \beta_J\big).$$

To prove this presentation, we introduce some properties of Poincaré duality algebras, which we need. An Artinian graded $\mathbb{R}$-algebra $A = A_0 \oplus A_1 \oplus \cdots \oplus A_s$ is said to be a **Poincaré duality algebra** (or Artinian Gorenstein algebra) of socle degree s if $A_s \cong \mathbb{R}$ and the map

$$A_i \times A_{s-i} \to A_s \cong \mathbb{R}, \quad (a, b) \mapsto ab$$

is non-degenerate. Such algebras are known to have nice symmetries. For example, the non-degenerateness tells an isomorphism of vector spaces

$$A_i \cong \mathrm{Hom}_{\mathbb{R}}(A_{s-i}, \mathbb{R}) \cong A_{s-i},$$

which guarantees that the Hilbert series

$$\mathrm{Hilb}(A, q) = \textstyle\sum_{k=0}^{s} \big(\dim_{\mathbb{R}} A_k\big) q^k$$

is palindromic. Poincaré duality algebras have a following nice relation with ideal quotients.

Lemma 5.7. *Let $f \in R$ be a homogeneous polynomial of degree d. Let $\mathcal{R}/\mathfrak{a}$ be a Poincaré duality algebra of socle degree s and $\mathcal{R}/\mathfrak{a}'$ a Poincaré duality algebra of socle degree $s-d$. If $\mathfrak{a}' \subset (\mathfrak{a} : f)$ and $f \notin \mathfrak{a}$, then $\mathfrak{a}' = (\mathfrak{a} : f)$.*

Proof. We note that the inclusion $\mathfrak{a}' \subset (\mathfrak{a} : f)$ tells that the multiplication map

$$\mathcal{R}/\mathfrak{a}' \xrightarrow{\times f} \mathcal{R}/\mathfrak{a}$$

is well-defined. To prove the assertion, we first prove that the map

$$\big(\mathcal{R}/\mathfrak{a}'\big)_{s-d} \xrightarrow{\times f} \big(\mathcal{R}/\mathfrak{a}\big)_s \tag{5.1}$$

is an isomorphism. To prove this, since $(\mathcal{R}/\mathfrak{a}')_{s-d} \cong (\mathcal{R}/\mathfrak{a})_s \cong \mathbb{R}$, it suffices to prove that the map is non-zero. Since $f + \mathfrak{a} \in (\mathcal{R}/\mathfrak{a})_d$ is non-zero by the assumption, the Poincaré duality of $\mathcal{R}/\mathfrak{a}$ tells that there is $g \in \mathcal{R}_{s-d}$ such that fg is nonzero in $(\mathcal{R}/\mathfrak{a})_s$. This guarantees that the map (5.1) is non-zero.

We now prove $\mathfrak{a}' = (\mathfrak{a} : f)$. Let g be a homogeneous polynomial of degree i such that $g \notin \mathfrak{a}'$. We prove $g \notin (\mathfrak{a} : f)$. By the Poincaré duality of $\mathcal{R}/\mathfrak{a}'$, there is an $h \in \mathcal{R}_{s-d-i}$ such that $gh + \mathfrak{a}' \in (\mathcal{R}/\mathfrak{a}')_{s-d}$ is nonzero. Then the injectivity of

(5.1) tells that fgh is nonzero in $\mathcal{R}/\mathfrak{a}$, in particular $gh \notin (\mathfrak{a} : f)$. This guarantees $g \notin (\mathfrak{a} : f)$ as desired. □

We also need the following fact on Artinian complete intersections (see Appendix for a proof).

Lemma 5.8. *If $A = \mathcal{R}/(f_1, \dots, f_n)$ is an Artinian complete intersection, then A is a Poincaré duality algebra of socle degree $\sum_{k=1}^{n}(\deg(f_k) - 1)$.*

We are now ready to prove Theorem 5.6.

Proof of Theorem 5.6. We first note that, by Theorem 4.13, Theorem 5.4(3) and Lemma 5.8, $\mathcal{R}/\mathfrak{a}(\mathcal{A}_J)$ is a Poincaré duality algebra of socle degree $|J|$ and $\mathcal{R}/(\mathcal{R}^W_+)$ is a Poincaré duality algebra of socle degree $|\Phi^+|$. We see in the proof of Theorem 5.4(2) that

$$\beta_J \mathfrak{a}(\mathcal{A}_J) \subset (P_1, \dots, P_n) = (\mathcal{R}^W_+)$$

and $\prod_{\alpha \in \Phi^+} \ell_\alpha \notin (\mathcal{R}^W_+)$. The former statement tells $\mathfrak{a}(\mathcal{A}_J) \subset \big((\mathcal{R}^W_+) : \beta_J\big)$ and the latter statement tells $\beta_J \notin (\mathcal{R}^W_+)$ since β_J divides $\prod_{\alpha \in \Phi^+} \ell_\alpha$. Then it follows from Lemma 5.7 that we have $\mathfrak{a}(\mathcal{A}_J) = \big((R^W_+) : \beta_J\big)$ as desired. □

Note. The idea of Solomon–Terao algebra comes from the work of [SoTe]. Theorem 5.6 appears in [AHMMS], and the name Solomon–Terao algebra appears in [AMMN]. Properties of Solomon–Terao alegebras are not well-understood. See [Abe] and [AMMN] for more details on these algebras.

Let $\mathcal{A}$ be a free arrangement with exponents $(d_1, \dots, d_n)$. If we substitute $q = 1$ in the Hilbert series of the Solomon–Terao algebra $\mathcal{R}/\mathfrak{a}(\mathcal{A})$ then we get

$$\mathrm{Hilb}(\mathcal{R}/\mathfrak{a}(\mathcal{A}), 1) = \prod_{k=1}^{n}(1 + d_k).$$

On the other hand, by Terao's factorization and Zaslabksy's theorem, we also have

$$\mathrm{Dist}_{\mathcal{A},R}(1) = |C(\mathcal{A})| = \prod_{k=1}^{n}(1 + d_k)$$

for any $R \in C(\mathcal{A})$. Comparing these equations, it would be natural to ask if, for any free arrangement $\mathcal{A}$, one has

$$\mathrm{Dist}_{\mathcal{A},R}(q) = \mathrm{Hilb}\big(\mathcal{R}/\mathfrak{a}(\mathcal{A}), q\big) = \prod_{k=1}^{n}(1 + q + \cdots + q^{d_k}) \tag{5.2}$$

for some $R \in C(\mathcal{A})$. The answer for this question is unfortunately no (see [BEZ]), but there are several classes of free arrangements such that we can indeed get equation (5.2). Supersolvable arrangements [BEZ] and inversion arrangements for rationally smooth elements (see [MOY, Sl15]) are instances of such arrangements, and we will later prove in section 7 that ideal arrangements have this property. It would be interesting to find a general condition of free arrangements which satisfy the equality (5.2) for some chamber R.

6. Torus equivariant cohomology of flag and Hessenberg varieties.

In this section and next section, we discuss a connection between Solomon–Terao algebra of ideal arrangements and cohomology rings of regular nilpotent Hessenberg varieties. Let G be a simple linear algebraic group of rank n, B a Borel subgroup of G, T a maximal torus in B, and $W = N(T)/T$ the Weyl group. A classical result of Borel [Bo] tells that the cohomology ring of the flag variety G/B is isomorphic to its coinvariant algebra $\mathcal{R}/(\mathcal{R}_+^W)$. Also, we see in the previous section that the coinvariant algebra $\mathcal{R}/(\mathcal{R}_+^W)$ equals to the Solomon–Terao algebra of the corresponding Weyl arrangement $\mathcal{A}_{\Phi^+}$, so we have an isomorphism

$$H^*(G/B) \cong \mathcal{R}/(\mathcal{R}_+^W) = \mathcal{R}/\mathfrak{a}(\mathcal{A}_{\Phi^+}). \tag{6.1}$$

The goal of sections 6 and 7 is to prove the following generalization[11] of the above isomorphism:

$$H^*(\mathrm{Hess}(\mathsf{N}, I)) \cong \mathcal{R}/\big((\mathcal{R}_+^W) : \beta_I\big) = \mathcal{R}/\mathfrak{a}(I). \tag{6.2}$$

To do this, we need some tools of torus equivariant cohomologies. The purpose of this chapter is to explain how algebra and combinatorics of hyperplane arrangements relate to cohomologies of Hessenberg varieties and it is out of scope of this chapter to explain details on equivariant cohomologies. Also, all necessary properties essentially appear in Chapter 9. So we quickly introduce known results on equivariant cohomologies of regular nilpotent Hessenberg varieties without proofs in this Section 6, and then in Section 7 we show how Solomon–Terao algebras relate to these equivariant cohomologies. For more information on equivariant cohomologies, see §2.2 in Chapter 9

We first explain a rough idea to prove (6.2). The flag variety G/B admit a natural left T-action. While a regular nilpotent Hessenberg variety $\mathrm{Hess}(\mathsf{N}, I) \subset G/B$ does not necessary admit this action, it admits an action of a 1-dimensional subtorus S of T with $(G/B)^T = (G/B)^S$ (see §6.2). Then we have the following inclusion

$$\begin{array}{ccc} G/B & \supset & (G/B)^T \\ & & \parallel \\ \cup & & (G/B)^S \\ & & \cup \\ \mathrm{Hess}(\mathsf{N}, I) & \supset & (\mathrm{Hess}(\mathsf{N}, I))^S. \end{array}$$

[11]The case when $I = \Phi^+$ is the isomorphism (6.1).

By the functoriality of equivariant cohomologies, the above diagram of inclusions induces the following commutative diagram of graded rings

$$
\begin{array}{ccc}
H^*_T(G/B) & \xrightarrow{\ \iota\ } & H^*_T\big((G/B)^T\big) \\
\downarrow & & \downarrow{\scriptstyle \pi} \\
H^*_S(G/B) & \xrightarrow{\ \iota'\ } & H^*_S\big((G/B)^S\big) \\
\downarrow & & \downarrow{\scriptstyle \mathsf{pr}_I} \\
H^*_S\big(\mathrm{Hess}(\mathsf{N},I)\big) & \xrightarrow{\ \iota_I\ } & H^*_S\big(\mathrm{Hess}(\mathsf{N},I)^S\big)
\end{array}
\tag{6.3}
$$

where $H_T(-)$ and $H_S(-)$ are T- and S-equivariant cohomologies, respectively. The key idea to prove (6.2) is to analyze the diagram (6.3) carefully. Indeed, we will see in this section that the image $\mathrm{Im}(\iota)$ of ι as well as maps π and pr_I have concrete algebraic descriptions. Then in the next section, using these descriptions, we give a presentation of $\mathrm{Im}(\mathsf{pr}_I \circ \pi \circ \iota)$ as a quotient of a polynomial ring and show that ι_I actually induces an isomorphism between $H^*_S(\mathrm{Hess}(\mathsf{N},I))$ and $\mathrm{Im}(\mathsf{pr}_I \circ \pi \circ \iota)$ by a purely algebraic argument. The isomorphism (6.2) on ordinal cohomologies then follows from this description of $H^*_S(\mathrm{Hess}(\mathsf{N},I)) \cong \mathrm{Im}(\mathsf{pr}_I \circ \pi \circ \iota)$.

6.1. **Equivariant cohomologies of flag varieties.** Let G be a simple linear algebraic group of rank n, B a Borel subgroup of G, T a maximal torus in B, and $W = N(T)/T$ the Weyl group. Let Φ, Φ^+, Φ^-, and $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be the set of roots, the set of positive roots, the set of negative roots, the set of simple roots associated with the above data. Let $V = \mathrm{span}_{\mathbb{R}}(\Phi)$ and $\mathcal{R} = \mathrm{sym}(V^*)$. The flag variety G/B admit a natural left T-action and it is known that the fixed point set $(G/B)^T$ of this action is given by

$$(G/B)^T = \bigsqcup_{w \in W} wB. \tag{6.4}$$

We will identify $(G/B)^T$ with the Weyl group W by the correspondence $wB \leftrightarrow w$.

Next, we list some properties of T-equivariant cohomologies of flag varieties which we need[12]. Recall that, for a topological space X with a continuous left T-action, the T-equivariant cohomology of X is $H^*_T(X) = H^*(ET \times_T X)$, where $ET \to BT$ is a universal principal T-bundle and $ET \times_T X$ is the orbit space of $ET \times X$ by the T-action $g \cdot (u, x) = (u \cdot g^{-1}, g \cdot x)$.

(T1) $H^*_T(\mathrm{pt}) = H^*(BT)$ is a polynomial ring in n variables, where pt is a point. More precisely, $H^*_T(\mathrm{pt}) = \mathbb{R}[\overline{\alpha}_1, \ldots, \overline{\alpha}_n]$ with $\overline{\alpha}_i = e^T(\mathbb{C}_{\alpha_i})$ the T-equivariant Euler class of the complex 1-dim T-module $\mathbb{C}_{\alpha_i}$ associated to $\alpha_i \in \Delta$[13]

[12]For (T1), see the beginning of Section 6 in Chapter 9. For (T2) and (T3), see e.g. Section 3 in Chapter 9 (only type A case is discussed there but they hold for any Lie type).

[13]Here roots are considered to be elements of the set of group homomorphism $\mathrm{Hom}(T, \mathbb{C}^*)$ which can be identified with $H^2(BT; \mathbb{Z})$ and are considered to live in the dual Lie algebra $\mathfrak{t}^*$ of the maximal compact torus in T. By taking an appropriate basis of V, we may assume that Φ is any set satisfying the definition of the root system in §3.1.

(T2) The inclusion $(G/B)^T \subset G/B$ induces an injection

$$H_T^*(G/B) \xrightarrow{\iota} H_T^*\big((G/B)^T\big) = \bigoplus_{w\in W} H_T^*(wB) \cong \bigoplus_{w\in W} \mathbb{R}[\overline{\alpha}_1,\dots,\overline{\alpha}_n].$$

For $f \in H_T^*((G/B)^T)$, we write $f|_w \in \mathbb{R}[\overline{\alpha}_1,\dots,\overline{\alpha}_n]$ for the element obtained from f by a natural projection $H_T^*((G/B)^T) \to H_T^*(wB) = \mathbb{R}[\overline{\alpha}_1,\dots,\overline{\alpha}_n]$. Also, let $\mathrm{ev} : \mathcal{R} \to \mathbb{R}[\overline{\alpha}_1,\dots,\overline{\alpha}_n]$ be the ring isomorphism given by $\mathrm{ev}(\ell_{\alpha_i}) = \overline{\alpha}_i$ (recall $\mathcal{R} = \mathbb{R}[\ell_{\alpha_1},\dots,\ell_{\alpha_n}]$).

(T3) There is a surjective ring homomorphism $\mathcal{R}[\overline{\alpha}_1,\dots,\overline{\alpha}_n]$ to $H_T^*(G/B)$ such that the composition

$$\phi : \mathcal{R}[\overline{\alpha}_1,\dots,\overline{\alpha}_n] \twoheadrightarrow H_T^*(G/B) \xrightarrow{\iota} H_T^*\big((G/B)^T\big)$$

satisfies
- $\phi(\ell_\alpha)|_w = \mathrm{ev}(\ell_{w(\alpha)})$ for $\alpha \in V \setminus \{\mathbf{0}\}$ and $w \in W$,
- $\phi(\overline{\alpha}_i)|_w = \overline{\alpha}_i$ for $i = 1,2,\dots,n$.

It is also known that the kernel of the map ϕ has the following concrete description:

$$\ker(\phi) = \big(f - \mathrm{ev}(f) \mid f \in \mathcal{R}_+^W\big).$$

Since ι is an injection, we have $\mathrm{Im}(\phi) \cong H_T^*(G/B)$, so the above (T3) gives a description of $H_T^*(G/B)$ as a submodule of $\bigoplus_{w\in W} \mathbb{R}[\overline{\alpha}_1,\dots,\overline{\alpha}_n]$. This description is often called the GKM presentation because of the seminal work of Goresky, Kottwitz, and MacPherson [GKM] on equivariant cohomologies (but we will not use the theory in this chapter). See §5 in Chapter 8 for more information on GKM theory.

Example 6.1. It is convenient to visualize an element $f \in H_T^*((G/B)^T)$ by writing the polynomial $f|_w$ on the chamber $C_w = w^{-1}(F)$. The pictures in Figure 13 are those corresponding to $\phi(x_1)$ and $\phi(x_2)$ for $\Phi_{B_2}^+ = \{\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_2, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2\}$. To compute these elements, use the right figure in Figure 6.

6.2. **Equivariant cohomology of regular nilpotent Hessenberg varieties.** The regular nilpotent Hessenberg variety $\mathrm{Hess}(\mathsf{N}, I)$ needs not admit an action of the torus T but it was shown by Harada and Tymoczko that $\mathrm{Hess}(\mathsf{N}, I)$ admit an action of a certain subgroup $S \cong \mathbb{C}^*$ of T. We quickly explain this.

Consider

$$T' = \{g \in T \mid \alpha_1(g) = \cdots = \alpha_n(g)\} \subset T,$$

where we regard simple roots $\alpha_1,\dots,\alpha_n$ as group maps from T to $\mathbb{C}^*$, and define S to be the identity component of T'. Let I be a lower ideal of Φ^+. By taking N as the sum of simple roots, S acts on $\mathrm{Hess}(\mathsf{N}, I)$ and the fixed point set by this action is given by the following simple form (note that we consider the identification $w \leftrightarrow wB$ in (6.4)). See [HT, Lemma 5.1 and Proposition 5.2] and §4 in Chapter 9 (for type A case).

Theorem 6.2 (Harada-Tymoczko)**.** *With the same notation as above,* $\mathrm{Hess}(\mathsf{N}, I)^S$ *equals to* $\mathrm{Sink}(I)$*, that is,*

$$\mathrm{Hess}(\mathsf{N}, I)^S = \{w \in W \mid w^{-1}(\Delta) \subset (-I) \cup \Phi^+\}.$$

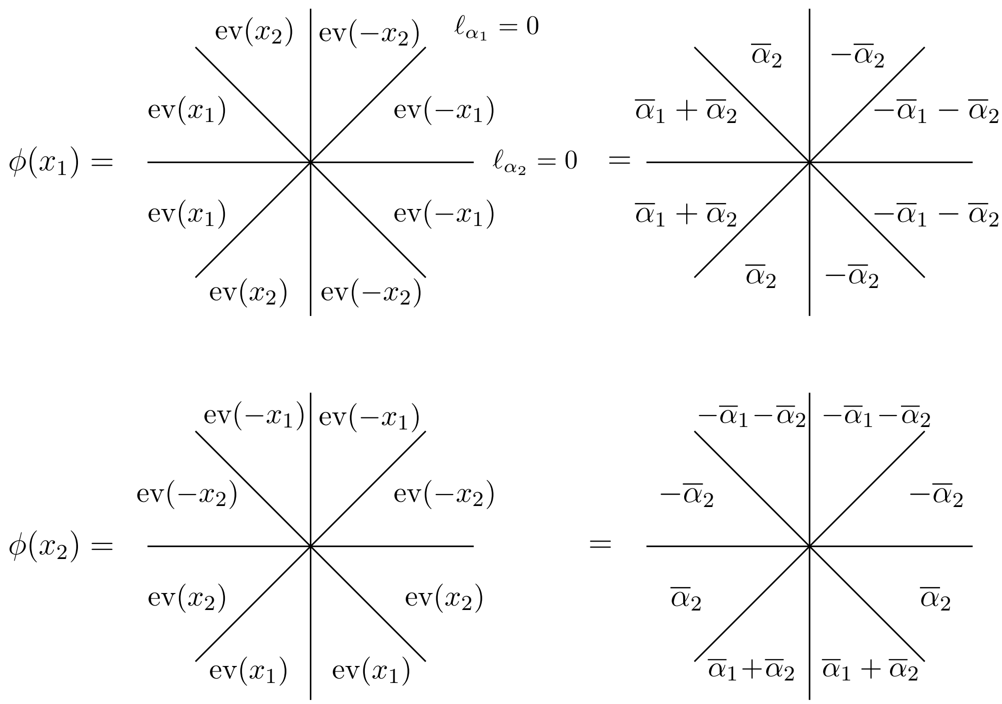


FIGURE 13. GKM presentation. Note that $\mathrm{ev}(x_1) = \overline{\alpha}_1 + \overline{\alpha}_2$ and $\mathrm{ev}(x_2) = \overline{\alpha}_2$ since $x_1 = \ell_{\alpha_1} + \ell_{\alpha_2}$ and $x_2 = \ell_{\alpha_2}$.

Below we list some properties of $H^*_S(\mathrm{Hess}(\mathsf{N}, I))$ which we need (see §2.2 and the beginning of §4 in Chapter 9 for more information).

(S1) $H^*_S(\mathrm{pt}) = H^*(BS)$ is a polynomial ring in one variable and can be written as $H^*_S(\mathrm{pt}) = \mathbb{R}[t]$ such that the homomorphism

$$H^*_T(\mathrm{pt}) = \mathbb{R}[\overline{\alpha}_1, \dots, \overline{\alpha}_n] \to \mathbb{R}[t] = H^*_S(\mathrm{pt})$$

induced by the containment $S \subset T$ is the map which send each $\overline{\alpha}_k$ to t.

(S2) $H^*_S(\mathrm{Hess}(\mathsf{N}, I))$ is a free $\mathbb{R}[t]$-module and

$$H^*\big(\mathrm{Hess}(\mathsf{N}, I)\big) \cong \big(H^*_S\big(\mathrm{Hess}(\mathsf{N}, I)\big)\big)/t\big(H^*_S\big(\mathrm{Hess}(\mathsf{N}, I)\big)\big).$$

(S3) The inclusion $\mathrm{Hess}(\mathsf{N}, I)^S \subset \mathrm{Hess}(\mathsf{N}, I)$ induces an injection

$$\iota : H^*_S\big(\mathrm{Hess}(\mathsf{N}, I)\big) \to H^*_S\big(\mathrm{Hess}(\mathsf{N}, I)^S\big) = \bigoplus_{w \in \mathrm{Sink}(I)} H^*_S(wB) \cong \bigoplus_{w \in \mathrm{Sink}(I)} \mathbb{R}[t].$$

As in §6.1, for $f \in H^*_S(\mathrm{Hess}(\mathsf{N}, I)^S)$ and $w \in \mathrm{Sink}(I)$, we write $f|_w \in \mathbb{R}[t]$ for the projection of f to $H^*_S(wB) = \mathbb{R}[t]$.

We now explain how properties listed above relate to the commutative diagram (6.3). As we exmpalined in the beginning of this sectionhave the following commutative diagram is induced from inclusions:

$$\begin{array}{ccccc} H_T^*(G/B) & \xrightarrow{\ \iota\ } & H_T^*\big((G/B)^T\big) & & (\cong \bigoplus_{w\in W} \mathbb{R}[\overline{\alpha}_1,\dots,\overline{\alpha}_n]) \\ \downarrow & & \downarrow \pi & & \\ H_S^*(G/B) & \xrightarrow{\ \iota'\ } & H_S^*\big((G/B)^S\big) & & (\cong \bigoplus_{w\in W} \mathbb{R}[t]) \\ \downarrow & & \downarrow \mathsf{pr}_I & & \\ H_S^*\big(\mathrm{Hess}(\mathsf{N},I)\big) & \xrightarrow{\ \iota_I\ } & H_S^*\big(\mathrm{Hess}(\mathsf{N},I)^S\big) & & (\cong \bigoplus_{w\in \mathrm{Sink}(I)} \mathbb{R}[t]), \end{array} \tag{6.5}$$

We now know that

- ι, ι', ι_I are injections (by (T2) an (S3)),
- ι has a concrete description explained in (T3),
- π is the map which send each $\overline{\alpha}_k$ to t (by (S1)), and
- pr_I is the natural projection with $\mathsf{pr}_I(f)|_w = f|_w$ for any $w \in \mathrm{Sink}(I)$,

where the last property follows from the functoriality.

7. Cohomology ring of $\mathrm{Hess}(\mathsf{N}, I)$ and ideal arrangements.

The goal of this section is to prove an isomorphism (6.2). As we explained in the previous section, we do this by analyzing the commutative diagram (6.5) algebraically.

We actually slightly generalize the situation. Indeed, we generalize the map pr_I in (6.5) to any subset $J \subset \Phi^+$. For a subset $J \subset \Phi^+$, let

$$\mathsf{pr}_J : \bigoplus_{w\in W} \mathbb{R}[t] \longrightarrow \bigoplus_{w\in \mathrm{Sink}(J)} \mathbb{R}[t]$$

be the projection defined by $\mathsf{pr}_J(f)|_w = f|_w$ for any $w \in \mathrm{Sink}(J)$, where $f|_w \in \mathbb{R}[t]$ denotes the wth component of $f \in \bigoplus_{w\in W} \mathbb{R}[t]$. When I is a lower ideal then this definition coincides with the map pr_I in (6.5). Also, we will identify $H_T^*((G/B)^T)$ and $H_S^*((G/B)^S)$ with $\bigoplus_{w\in W} \mathbb{R}[\overline{\alpha}_1,\dots,\overline{\alpha}_n]$ and $\bigoplus_{w\in W} \mathbb{R}[t]$, respectively, and regard each object in the second column of (6.5) as a direct sums of a polynomial ring.

7.1. **GKM presentation in one variable.** We first study the image of $\pi \circ \iota$ in the commutative diagram (6.5). We note that this image is obtained from the GKM representaion of $H_T^*(G/B)$ by substituting each $\overline{\alpha}_k$ (which corresponds to a simple root) with the variable t.

Recall that ϕ is the composition

$$\phi : \mathcal{R}[\overline{\alpha}_1,\dots,\overline{\alpha}_n] \twoheadrightarrow H_T^*(G/B) \xrightarrow{\ \iota\ } H_T^*\big((G/B)^T\big).$$

(Recall that we identify $H^*_T((G/B)^T)$ and $\bigoplus_{w\in W}\mathbb{R}[\overline{\alpha}_1,\dots,\overline{\alpha}_n]$.) Consider the following commutative diagram

$$\begin{array}{ccc} \mathcal{R}[\overline{\alpha}_1,\dots,\overline{\alpha}_n] & \xrightarrow{\ \phi\ } & \bigoplus_{w\in W}\mathbb{R}[\overline{\alpha}_1,\dots,\overline{\alpha}_n] \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ \mathcal{R}[t] & \xrightarrow{\ \phi'\ } & \bigoplus_{w\in W}\mathbb{R}[t], \end{array} \tag{7.1}$$

where ϕ' is the map defined by $\phi'(f)|_w = \pi(\phi(f))|_w$ for any $f\in\mathcal{R}$ and $\phi'(t)|_w = t$ for any $w\in W$. We note that the image of $\pi\circ\phi$ in (7.1) is nothing but the image of $\pi\circ\iota$ in (6.5). Also, since vertical maps in (7.1) are surjective, we have

$$\operatorname{Im}(\pi\circ\iota)=\operatorname{Im}(\pi\circ\phi)=\operatorname{Im}(\phi'\circ\pi)=\operatorname{Im}(\phi'). \tag{7.2}$$

So we can analyze $\operatorname{Im}(\pi\circ\iota)$ by looking the image of ϕ'. Below we list some properties of ϕ'.

Lemma 7.1. *Let $\alpha\in\Phi$ and $\mathcal{J}=(f-\pi(\operatorname{ev}(f))\mid f\in\mathcal{R}^W_+)\subset\mathcal{R}[t]$.*

1. $\operatorname{Ker}(\phi')=\mathcal{J}$.
2. $\phi'(\ell_\alpha)|_w=\operatorname{ht}(w(\alpha))\cdot t$ *for any* $w\in W$.
3. $\phi'(\ell_\alpha+t)|_w=0$ *if and only if* $w(\alpha)\in-\Delta$.

Proof. Recall $\operatorname{Ker}(\phi)=(f-\operatorname{ev}(f)\mid f\in\mathcal{R}^W_+)$ and $H^*_T(G/B)\cong\mathcal{R}[\overline{\alpha}_1,\dots,\overline{\alpha}_n]/\operatorname{Ker}(\phi)$. Then since

$$\operatorname{Im}(\phi')=\operatorname{Im}(\iota')\cong H^*_S(G/B)\cong H^*_T(G/B)\otimes_{\mathbb{R}[\overline{\alpha}_1,\dots,\overline{\alpha}_n]}\big(\mathbb{R}[\overline{\alpha}_1,\dots,\overline{\alpha}_n]/(\overline{\alpha}_i-\overline{\alpha}_j\mid i\neq j)\big),$$

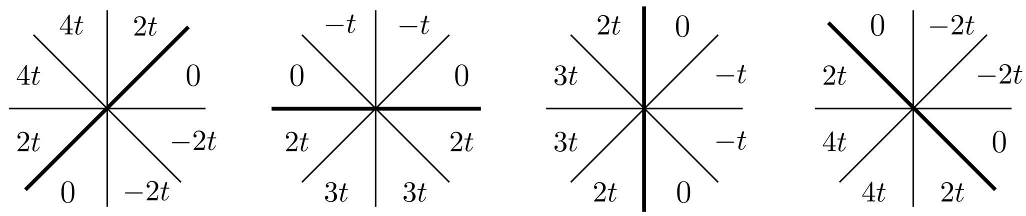
it follows that $\operatorname{Im}(\phi')\cong\mathcal{R}[t]/\mathcal{J}$. Since $\operatorname{Ker}(\phi')\supset\mathcal{J}$ is clear, we have $\operatorname{Ker}(\phi')=\mathcal{J}$, as desired.

The assertion (2) follows from the fact that $\pi(\operatorname{ev}(\ell_\gamma))=\operatorname{ht}(\gamma)\cdot t$ for any $\gamma\in\Phi^+$. Also, the last assertion follows from (2) and the fact that for any $\gamma\in\Phi$ one has $\operatorname{ht}(\gamma)=-1$ if and only if $\gamma\in-\Delta$. □

We explain a bit more about the meaning of the third statement of the above lemma. Recall that Lemma 3.6 tells that we have $w(\alpha)\in-\Delta\Leftrightarrow\alpha\in w^{-1}(-\Delta)$ if and only if the hyperplane H_α is adjacent to the Weyl chamber C_w and separates the fundamental chamber F and C_w. Lemma 7.1(3) tells that if we visualize $\pi(\ell_\alpha+t)$ on the Weyl arrangement $\mathcal{A}_{\Phi^+}$ then zero appears exactly on chambers C_w such that H_α supports a facet of $\overline{C}_w$ and H_α separates F and C_w. See Figure 14, where we visualize $\phi'(x_1-x_2+t),\phi'(x_2+t),\phi'(x_1+t)$ and $\phi'(x_1+x_2+t)$ for $\Phi^+_{B_2}=\{\mathbf{e}_1-\mathbf{e}_2,\mathbf{e}_2,\mathbf{e}_1,\mathbf{e}_1+\mathbf{e}_2\}$. (See also Figure 13 for the computation of $\phi(x_1)$ and $\phi(x_2)$.)

7.2. **Restriction to the sink.** By (7.2) and Lemma 7.1(1), we can see $\operatorname{Im}(\pi\circ\iota)=\mathcal{R}[t]/\mathcal{J}$. We next study the image of $\mathsf{pr}_J\circ\pi\circ\iota$. We consider the composition

$$\mathcal{R}[t]\xrightarrow{\phi'}\bigoplus_{w\in W}\mathbb{R}[t]\xrightarrow{\mathsf{pr}_J}\bigoplus_{w\in\operatorname{Sink}(J)}\mathbb{R}[t].$$



$\phi'(x_1 - x_2 + t)$ $\qquad$ $\phi'(x_2 + t)$ $\qquad$ $\phi'(x_1 + t)$ $\qquad$ $\phi'(x_1 + x_2 + t)$

FIGURE 14. Visualizations of $\pi(\ell_\alpha + t)$.

Theorem 7.2. *Let $J \subset \Phi^+$ and $\widetilde{\beta_J} = \prod_{\alpha \in \Phi^+ \setminus J}(\ell_\alpha + t)$. Then we have* $\mathrm{Ker}(\phi' \circ \mathsf{pr}_J) = (\mathcal{J} : \widetilde{\beta}_J)$. *In particular, we have*

$$\mathrm{Im}(\mathsf{pr}_J \circ \pi \circ \iota) = \mathrm{Im}(\mathsf{pr}_J \circ \phi') \cong \mathcal{R}[t]/\big(\mathcal{J} : \widetilde{\beta}_J\big).$$

Proof. For two elements $f, g \in \bigoplus_{w \in W} \mathbb{R}[t]$, we define their product fg by setting $fg|_w = f|_w g|_w$ for any $w \in W$. Note that $\phi'(ab) = \phi'(a)\phi'(b)$ for $a, b \in \mathcal{R}[t]$. Let

$$N = \left\{ f \in \bigoplus_{w \in W} \mathbb{R}[t] \mid f|_w = 0 \text{ for any } w \in \mathrm{Sink}(J) \right\}.$$

Then, by the definition of pr_J, for any $f \in \mathcal{R}[t]$, we clearly have

$$\mathsf{pr}_J \circ \phi'(f) = 0 \quad \Leftrightarrow \quad \phi'(f) \in N. \tag{7.3}$$

Recall $\mathrm{Sink}(J) = \{w \in W \mid w^{-1}(\Delta) \subset (-J) \cup \Phi^+\}$. Then by Lemma 7.1(3)

$$\begin{aligned} \pi(\widetilde{\beta}_J)|_w = 0 &\Leftrightarrow w(\alpha) \in -\Delta \text{ for some } \alpha \in \Phi^+ \setminus J \\ &\Leftrightarrow -\alpha \in w^{-1}(\Delta) \text{ for some } \alpha \in \Phi^+ \setminus J \\ &\Leftrightarrow w \in W \setminus \mathrm{Sink}(J). \end{aligned}$$

This tells that, since $\mathbb{R}[t]$ is a domain, one has

$$h \in N \Leftrightarrow h|_w = 0 \text{ for all } w \in \mathrm{Sink}(J) \Leftrightarrow \phi'(\widetilde{\beta}_J)h = 0. \tag{7.4}$$

Now (7.3) and (7.4) show, for any $f \in \mathcal{R}[t]$, we have

$$\mathsf{pr}_J \circ \phi'(f) = 0 \Leftrightarrow \phi'(\widetilde{\beta}_J f) = 0 \Leftrightarrow \widetilde{\beta}_J f \in \mathcal{J} \Leftrightarrow f \in (\mathcal{J} : \widetilde{\beta}_J),$$

where the second equivalence follows from Lemma 7.1(1). This proves the desired equality $\mathrm{Ker}(\mathsf{pr}_J \circ \phi') = (\mathcal{J} : \widetilde{\beta}_J)$. □

Example 7.3. Let $I = \{\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_2, \mathbf{e}_1\} \subset \Phi^+_{B_2}$. Then, since $\mathrm{ev}(x_1) = \overline{\alpha}_1 + \overline{\alpha}_2$ and $\mathrm{ev}(x_2) = \overline{\alpha}_2$, we have

$$\mathcal{J} = (x_1^2 + x_2^2 - 5t^2, x_1^4 + x_2^4 - 17t^4)$$

and

$$\begin{aligned} \mathrm{Im}(\mathsf{pr}_J \circ \phi') &\cong \mathbb{R}[x_1, x_2, t]/\big(\mathcal{J} : (x_1 + x_2 + t)\big) \\ &= \mathbb{R}[x_1, x_2, t]/(x_1^2 + x_2^2 - 5t^2, x_1^2(x_1 - x_2) + x_1x_2t + 2x_2t^2 - 3x_1t^2 - 2t^3). \end{aligned}$$

7.3. **What happens if we substitute $t = 0$?** For an ideal $\mathfrak{a} \subset \mathcal{R}[t]$, we write $\mathfrak{a}|_{t=0}$ for the ideal of $\mathcal{R}$ obtained from $\mathfrak{a}$ by the substitution $t = 0$, that is,

$$\mathfrak{a}|_{t=0} := \{f(0) \mid f(t) \in \mathfrak{a}\} \subset \mathcal{R}.$$

For example, by the definition of $\mathcal{J}$, we have $\mathcal{J}|_{t=0} = (\mathcal{R}_+^W)$. In the previous subsections, we see that $\mathrm{Im}(\pi \circ \iota) \cong \mathcal{R}[t]/\mathcal{J}$ and $\mathrm{Im}(\mathsf{pr}_J \circ \pi \circ \iota) \cong \mathcal{R}[t]/(\mathcal{J} : \widetilde{\beta}_J)$. In this subsection, we study when the equality $(\mathcal{J} : \widetilde{\beta}_J)|_{t=0} = ((\mathcal{R}_+^W) : \beta_J)$ holds. Our goal is to show that if $\mathcal{A}_J$ is free and coconvex then we get the equality $(\mathcal{J} : \widetilde{\beta}_J)|_{t=0} = ((\mathcal{R}_+^W) : \beta_J)$. This will be used in the next subsection to obtain a structure of ordinal cohomology from an S-equivariant cohomology of regular nilpotent Hessenberg variety.

We recall a few basic facts on (graded) free $\mathbb{R}[t]$-modules. The next fact is an immediate consequence of the fact that $\mathbb{R}[t]$ is a PID.

Lemma 7.4. *Any submodule of a free $\mathbb{R}[t]$-module is a free $\mathbb{R}[t]$-module.*

Lemma 7.5. *Let $F = \bigoplus_{k=1}^m \mathbb{R}[t]e_k$ be a free $\mathbb{R}[t]$-module of rank m with a free $\mathbb{R}[t]$-basis $e_1, \dots, e_m$ and $F' = \bigoplus_{k=1}^l \mathbb{R}[t]e_k$ with $l \leq m$. Consider the natural projection $p : F \to F'$ given by $p(\sum_{k=1}^m f_k e_k) = \sum_{k=1}^l f_k e_k$. If a submodule M of F has rank m then $p(M)$ has rank l.*

Proof. Let $\mathcal{Q} = \mathbb{R}(t)$ be the field of fractions of $\mathbb{R}[t]$. Then $\dim_{\mathcal{Q}}(M \otimes \mathcal{Q}) = \mathrm{rank}(M) = m$ so $M \otimes \mathcal{Q} = F \otimes \mathcal{Q}$. This tells $p(M) \otimes \mathcal{Q} = F' \otimes \mathcal{Q}$ has $\mathcal{Q}$-dimension l, so $p(M)$ has rank l. □

If F is a graded free $\mathbb{R}[t]$-module, then as $\mathbb{R}$-vector spaces one has

$$F \cong (F/tF) \otimes_{\mathbb{R}} \mathbb{R}[t].$$

Since $\mathrm{Hilb}(\mathbb{R}[t], q) = \frac{1}{1-q}$ we have the following lemma.

Lemma 7.6. *If F is a free $\mathbb{R}[t]$-module, then* $\mathrm{Hilb}(F, q) = \frac{1}{1-q}\mathrm{Hilb}(F/tF, q)$.

If $\mathfrak{a}$ is an ideal of $\mathcal{R}[t]$, then we have a natural isomorphism

$$(\mathcal{R}[t]/\mathfrak{a})/\big(t(\mathcal{R}[t]/\mathfrak{a})\big) \cong \mathcal{R}[t]/\big(\mathfrak{a} + (t)\big) \cong \mathcal{R}/(\mathfrak{a}|_{t=0}).$$

Thus Lemma 7.6 implies the following fact.

Corollary 7.7. *Let $\mathfrak{a} \subset \mathcal{R}[t]$ be an ideal. If $\mathcal{R}[t]/\mathfrak{a}$ is a free $\mathbb{R}[t]$-module, then*

$$\mathrm{Hilb}(\mathbb{R}[t]/\mathfrak{a}, q) = \frac{1}{1-q}\mathrm{Hilb}(\mathcal{R}/(\mathfrak{a}|_{t=0}), q).$$

In particular, the rank of $\mathcal{R}[t]/\mathfrak{a}$ equals to $\dim_{\mathbb{R}}(\mathcal{R}/(\mathfrak{a}|_{t=0}))$.

Theorem 7.8. *Let $J \subset \Phi^+$.*

(1) *The ring $\mathcal{R}[t]/(\mathcal{J} : \widetilde{\beta}_J)$ is a free $\mathbb{R}[t]$-module.*
(2) *If $\mathcal{A}_J$ is a free arrangement and J is coconvex then*

$$\big(\mathcal{J} : \widetilde{\beta}_J\big)|_{t=0} = \big((\mathcal{R}_+^W) : \beta_J\big).$$

Proof. (1) Since $\mathrm{Im}(\mathsf{pr}_J \circ \phi')$ is a submodule of a free $\mathbb{R}[t]$-module $\bigoplus_{w \in \mathrm{Sink}(J)} \mathbb{R}[t]$, the statement follows from Lemma 7.4 and the isomorphism $\mathcal{R}[t]/(\mathcal{J} : \widetilde{\beta}_J) \cong \mathrm{Im}(\mathsf{pr}_J \circ \phi')$ in Theorem 7.2.

(2) Suppose that $\mathcal{A}_J$ is free and J is coconvex. Let $\mathfrak{n}_J = \big(\mathcal{J} : \widetilde{\beta}_J)|_{t=0}$. We first prove $\big((\mathcal{R}^W_+) : \beta_J\big) \supset \mathfrak{n}_J$. Suppose $f \in \mathfrak{n}_J$. Then there is $g = g(t) \in \mathcal{R}[t]$ such that $g(0) = f$ and $\widetilde{\beta}_J g \in \mathcal{J}$. Let $h(t) = \widetilde{\beta}_J g$. Since $\mathcal{J}|_{t=0} = (\mathcal{R}^W_+)$, it follows that $\beta_J f = h(0) \in (\mathcal{R}^W_+)$, telling that $f \in \big((\mathcal{R}^W_+) : \beta_J\big)$. This proves $\big((\mathcal{R}^W_+) : \beta_J\big) \supset \mathfrak{n}_J$.

Now, to prove $\mathfrak{n}_J = \big((\mathcal{R}^W_+) : \beta_J\big)$, it suffices to prove

$$\dim_{\mathbb{R}}(\mathcal{R}/\mathfrak{n}_J) = \dim_{\mathbb{R}}\big(\mathcal{R}/\big((\mathcal{R}^W_+) : \beta_J\big)\big). \tag{7.5}$$

Since $\mathcal{A}_J$ is free, we know by Theorem 5.4(4) and Theorem 5.6 that

$$\dim_{\mathbb{R}}\big(\mathcal{R}/\big((\mathcal{R}^W_+) : \beta_J\big)\big) = \dim_{\mathbb{R}}\big(\mathcal{R}/\mathfrak{a}(\mathcal{A}_J)\big) = |C(\mathcal{A}_J)|.$$

On the other hand, since $\mathcal{R}[t]/(\mathcal{J} : \widetilde{\beta}_J)$ is a free $\mathbb{R}[t]$-module, $\dim_{\mathbb{R}}(\mathcal{R}/\mathfrak{n}_J)$ equals to the rank of $\mathcal{R}[t]/(\mathcal{J} : \widetilde{\beta}_J) \cong \mathrm{Im}(\mathsf{pr}_J \circ \phi')$ as a free $\mathbb{R}[t]$-module by Corollary 7.7, which equals to $|\mathrm{Sink}(J)|$ by Lemma 7.5 (as $\mathrm{Im}(\phi) \cong H^*_T(G/B)$ has rank $|W|$ and $\mathrm{Im}(\phi') = \mathrm{Im}(\pi \circ \phi)$). Since J is coconvex we have $|\mathrm{Sink}(J)| = |C(\mathcal{A}_J)|$ by Theorem 3.13, so we get the desired equation (7.5). □

7.4. **Cohomology rings of regular nilpotent Hessenberg varieties.** We are now ready to prove main results in this section. Our first goal is to give the following description of the S-equivariant cohomology of $\mathrm{Hess}(\mathsf{N}, I)$.

Theorem 7.9. *For a lower ideal $I \subset \Phi^+$, we have an isomorphism of rings*

$$H^*_S\big(\mathrm{Hess}(\mathsf{N}, I)\big) \cong \mathcal{R}[t]/(\mathcal{J} : \widetilde{\beta}_J).$$

We need one technical statement on formal power series.

Lemma 7.10. *Let $f(q) = \sum_{k=0}^d a_k q^k$ and $g(q) = \sum_{k=0}^d b_k q^k$ be palindromic polynomials of degree d with $\sum_{k=0}^d a_k = \sum_{k=0}^d b_k$. If we have a coefficient-wise inequality of formal power series $\frac{1}{1-q} f(q) \leq \frac{1}{1-q} g(q)$ then $f(q) = g(q)$.*

Proof. Let $c = \sum_{k=0}^d a_k = \sum_{k=0}^d b_k$. Since the coefficient of q^i in $\frac{1}{1-q} f(q)$ is $\sum_{k=0}^i a_k$, the inequality $\frac{1}{1-q} f(q) \leq \frac{1}{1-q} g(q)$ tells

$$\textstyle\sum_{k=0}^i a_k \leq \sum_{k=0}^i b_k \quad \text{for all } i.$$

But we also have

$$\textstyle\sum_{k=0}^i a_k = c - \sum_{k=0}^{d-i-1} a_k \geq c - \sum_{k=0}^{d-i-1} b_k = \sum_{k=0}^i b_k$$

for all i, where we use the palindromicity $a_k = a_{d-k}$ and $b_k = b_{d-k}$ in the first and the third equality. These prove $\sum_{k=0}^i a_k = \sum_{k=0}^i b_k$ for all i, equivalently, $a_i = b_i$ for all i. □

Proof of Theorem 7.9. Let A_I be the image of the injection

$$\iota_I : H^*_S\big(\mathrm{Hess}(\mathsf{N}, I)\big) \longrightarrow H^*_S(\mathrm{Hess}(\mathsf{N}, I)^S)$$

in the diagram (6.5). The commutativity of the diagram (6.5) tells that the image of $\mathsf{pr}_I \circ \pi \circ \iota$ is contained in A_I and we see in Theorem 7.2 that

$$\mathrm{Im}(\mathsf{pr}_I \circ \pi \circ \iota) \cong \mathcal{R}[t]/(\mathcal{J} : \widetilde{\beta}_I).$$

Then, to prove the statement, it suffices to prove that A_I and $\mathcal{R}[t]/(\mathcal{J} : \widetilde{\beta}_J)$ have the same Hilbert series.

By the property (S2), Lemma 7.6 and Theorem 3.15, we have

$$\mathrm{Hilb}(A_I, q) = \frac{1}{1-q}\mathrm{Poin}\big(\mathrm{Hess}(\mathsf{N}, I), \sqrt{q}\big) = \frac{1}{1-q}\mathrm{Dist}_{\mathcal{A}_I, R_0}(q).$$

(For $\mathrm{Hilb}(-, q)$ we are considering that each x_i has degree 1, while for $\mathrm{Poin}(-, q)$ we are considering that each x_i has degree 2.) On the other hand, since $\mathcal{A}_I$ is a free arrangement and I is coconvex, by Corollary 7.7 and Theorem 7.8 we have

$$\mathrm{Hilb}\big(\mathcal{R}[t]/(\mathcal{J} : \widetilde{\beta}_I), q\big) = \frac{1}{1-q}\mathrm{Hilb}\big(\mathcal{R}/\big((\mathcal{R}^W_+) : \beta_I\big), q\big) = \frac{1}{1-q}\mathrm{Hilb}(\mathcal{R}/\mathfrak{a}(\mathcal{A}_I), q).$$

Then, since

$$\mathrm{Dist}_{\mathcal{A}_I, R_0}(1) = |C(\mathcal{A}_I)| = \mathrm{Hilb}(\mathcal{R}/\mathfrak{a}(\mathcal{A}_I), 1),$$

where we use Theorem 5.4(4) for the second equation, and since both $\mathrm{Dist}_{\mathcal{A}_I, R_0}(q)$ and $\mathrm{Hilb}(\mathcal{R}/\mathfrak{a}(\mathcal{A}_I), q)$ are palindromic polynomials of degree $|\mathcal{A}_I| = |I|$ (by Lemma 2.9 and Theorem 5.4(3)), it follows from Lemma 7.10 that A_I and $\mathcal{R}[t]/(\mathcal{J} : \widetilde{\beta}_I)$ have the same Hilbert series as desired. □

We now prove the isomorphism (6.2).

Theorem 7.11 (Abe-Horiguchi-Masuda-Murai-Sato)**.** *Let $I \subset \Phi^+$ be a lower ideal. We have an isomorphism of rings*

$$H^*\big(\mathrm{Hess}(\mathsf{N}, I)\big) \cong \mathcal{R}/\mathfrak{a}(\mathcal{A}_I).$$

Proof. By the previous theorem and the property (S2), we have

$$\begin{aligned} H^*(\mathrm{Hess}(\mathsf{N}, I)) &\cong \big(H^*_S\big(\mathrm{Hess}(\mathsf{N}, I)\big)\big)/t\big(H^*_S\big(\mathrm{Hess}(\mathsf{N}, I)\big)\big) \\ &\cong \big(\mathcal{R}[t]/(\mathcal{J} : \widetilde{\beta}_I)\big)/t\big(\mathcal{R}[t]/(\mathcal{J} : \widetilde{\beta}_I)\big) \\ &\cong \mathcal{R}/\big((\mathcal{J} : \widetilde{\beta}_I)|_{t=0}\big). \end{aligned}$$

But the last ring in the above equation is isomorphic to $\mathcal{R}/((\mathcal{R}^W_+) : \beta_I) = \mathcal{R}/\mathfrak{a}(\mathcal{A}_I)$ by Theorems 5.6 and 7.8, as desired. □

Example 7.12. Consider the lower ideal $I = \{\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_2, \mathbf{e}_1\} \subset \Phi^+_{B_2}$. The computation in Example 7.3 tells that $H^*_S(\mathrm{Hess}(\mathsf{N}, I))$ is isomorphic to

$$\mathbb{R}[x_1, x_2, t]/(x_1^2 + x_2^2 - 5t^2, x_1^2(x_1 - x_2) + x_1x_2t + 2x_2t^2 - 3x_1t^2 - 2t^3).$$

In particular, we get

$$H^*\big(\mathrm{Hess}(\mathsf{N}, I)\big) \cong \mathbb{R}[x_1, x_2]/(x_1^2 + x_2^2, x_1^2(x_1 - x_2)).$$

The above computation is based on the computation of the colon ideal, but in general the computation of a colon ideal is not easy. In the next section we will see a better way to compute $\mathcal{R}/\mathfrak{a}(\mathcal{A}_I)$.

7.5. **Two interesting factorization formula.** We give two quick applications of Theorem 7.11. Consider the isomorphism $H^*(G/B) \cong \mathcal{R}/(\mathcal{R}^W_+)$. Since $\mathcal{R}/(\mathcal{R}^W_+) = \mathcal{R}/(P_1, \dots, P_n)$ is a complete intersection, by the formula of Hilbert series of complete intersections, we have the following factorization of the Poincaré series

$$\mathrm{Poin}(G/B, q) = \mathrm{Hilb}(\mathcal{R}/(P_1, \dots, P_n), q^2) = \prod_{k=1}^{n}(1 + q^2 + q^4 + \cdots + q^{2m_k}),$$

where $m_1 = \deg P_1 - 1, \dots, m_n = \deg P_n - 1$ are the exponents of the Weyl group W. Theorem 7.11 tells that this simple factorization formula of Poincaré series can be extended to arbitrary regular nilpotent Hessenberg variety.

Let $I \subset \Phi^+$ be a lower ideal and let $d(I) = (d^I_1, \dots, d^I_n)$ be the ideal exponents of I (that is, exponents of $\mathcal{A}_I$). Then, by Theorem 4.13, $\mathcal{D}(\mathcal{A}_I)$ is a free module with a free $\mathcal{R}$-basis $\theta_1, \dots, \theta_n$ with $\deg \theta_i = d^I_i$ for $i = 1, 2, \dots, n$, and the Solomon–Terao algebra is a complete intersection of the form

$$\mathcal{R}/\mathfrak{a}(\mathcal{A}_I) = \mathcal{R}/(f_1, \dots, f_n)$$

with $\deg f_i = \deg \eta(\theta_i) = d^I_i + 1$ for $i = 1, 2, \dots, n$. Then by Theorem 7.11 we have

$$\mathrm{Poin}\big(\mathrm{Hess}(\mathsf{N}, I), q\big) = \mathrm{Hilb}\big(\mathcal{R}/(f_1, \dots, f_n), q^2\big) = \prod_{k=1}^{n}(1 + q^2 + q^4 + \cdots + q^{2d^I_k}),$$

where we use the factorization formula of Hilbert series of complete intersections. To summarize, we get the following consequence.

Corollary 7.13. *Let $I \subset \Phi^+$ be a lower ideal and $d(I) = (d^I_1, \dots, d^I_n)$ the ideal exponents of I. Then*

$$\mathrm{Poin}\big(\mathrm{Hess}(\mathsf{N}, I), q\big) = \prod_{k=1}^{n}(1 + q^2 + \cdots + q^{2d^I_k}).$$

Example 7.14. Consider the ideal

$$I = \{\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_1 - \mathbf{e}_3, \mathbf{e}_1, \mathbf{e}_2 - \mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_3\} \subset \Phi^+_{B_3}.$$

Then $d(I) = (3, 3, 1)$ and Corollary 7.13 tells

$$\mathrm{Poin}(\mathrm{Hess}(\mathsf{N}, I), q) = (1 + q^2 + q^4 + q^6)(1 + q^2 + q^4 + q^6)(1 + q^2).$$

The above factorization formula has a combinatorial aspect. As we saw in Theorem 3.15, the Poincaré series of the Hessenberg variety $\mathrm{Hess}(\mathsf{N}, I)$ can be identified with a distance enumerator polynomial of the ideal arrangement $\mathcal{A}_I$. Thus Corollary 7.13 also gives the following factorization formula on distance enumerator polynomials.

Corollary 7.15. *Let $I \subset \Phi^+$ be a lower ideal, $d(I) = (d^I_1, \dots, d^I_n)$ the ideal exponents of I, and $R_0 \in C(\mathcal{A}_I)$ the unique chamber of $\mathcal{A}_I$ that contains the fundamental Weyl chamber. Then*

$$\mathrm{Dist}_{\mathcal{A}_I, R_0}(q) = \prod_{k=1}^{n}(1 + q + \cdots + q^{d^I_k}).$$

Note. Theorem 7.2 essentially appears in [AHMMS], where it is assumed that J is a lower ideal. We do not know if this algebra has a nice meaning even when J is not a lower ideal.

The proof of Theorem 7.11 is based on the isomorphisms

$$\mathcal{R}/\mathfrak{a}(\mathcal{A}_I) = \mathcal{R}/\big((\mathcal{R}_+^W) : \beta_I\big)$$

and

$$H_S^*\big(\mathrm{Hess}(\mathsf{N}, I)\big) \cong \mathrm{Im}(\mathsf{pr}_I \circ \pi \circ \iota),$$

where $\iota, \pi, \mathsf{pr}_I$ are maps in (6.3). As you may notice, this proof does not give a direct connection between $\mathcal{D}(\mathcal{A}_I)$ and $\mathrm{Hess}(\mathsf{N}, I)$. It will be interesting to find such a direct connection.

The formula in Corollaries 7.13 and 7.15 was conjectured by Sommers and Tymoczko, who proved it for types A, B, C, G_2, E_4 and F_6 [SoTy, Theorem 4.1]. Later the formula was also proved for type D_4, D_5, D_6, D_7, E_7 and for most lower ideals of E_8 by Schauenburg and Röhrel (see [Röh, Theorem 1.28]). But a combinatorial proof of the formula for arbitrary Lie type is not known.

8. Explicit description of $\mathcal{D}(\mathcal{A}_I)$ and $H^*(\mathrm{Hess}(\mathsf{N}, I))$.

In the previous section, we see the isomorphism

$$\mathcal{R}/\mathfrak{a}(\mathcal{A}_I) \cong H^*(\mathrm{Hess}(\mathsf{N}, I)).$$

One nice application of this isomorphism is that we can explicitly compute generators of the ideal $\mathfrak{a}(\mathcal{A}_I)$ by computing a basis of $\mathcal{D}(\mathcal{A}_I)$ and such a computation yields an explicit presentation of $H^*(\mathrm{Hess}(\mathsf{N}, I))$ as a quotient of a polynomial ring. In this last section, we give such an explicit presentation for type A_{n-1}, B_n, C_n and D_n based on the computations given in [AHMMS, §10] and [EHNT].

Throughout the rest of this section, we set $V = \mathbb{R}^n$ and $\mathcal{R} = \mathrm{sym}(V^*) = \mathbb{R}[x_1, \dots, x_n]$, and consider the set of roots $\Phi^+_{A_{n-1}}, \Phi^+_{B_n}, \Phi^+_{C_n}, \Phi^+_{D_n}$ given in §3.1.

8.1. Type A_{n-1} and B_n.

Recall that

$$\Phi^+_{A_{n-1}} = \{\mathbf{e}_i - \mathbf{e}_j \mid 1 \le i < j \le n\}$$

and

$$\Phi^+_{B_n} = \{\mathbf{e}_i \pm \mathbf{e}_j \mid 1 \le i < j \le n\} \cup \{\mathbf{e}_i \mid i = 1, 2, \dots, n\},$$

where we consider that $\Phi^+_{A_{n-1}}$ is a subset of $\mathbb{R}^n$ so that lower ideals in $\Phi^+_{A_{n-1}}$ can be considered as lower ideals in $\Phi^+_{B_n}$. We will give a basis of $\mathcal{D}(\mathcal{A}_I)$ for any lower ideal I in $\Phi^+_{B_n}$.

We first introduce a few notation. For $1 \le i < j \le 2n - i + 1$, we set

$$\alpha_{i,j} = \begin{cases} \mathbf{e}_i - \mathbf{e}_j & (i < j \le n), \\ \mathbf{e}_i & (j = n+1), \\ \mathbf{e}_i + \mathbf{e}_{2n+2-j} & (n+2 \le j \le 2n - i + 1), \end{cases}$$

and

$$\mathcal{H}_i = \{\alpha_{i,i+1}, \alpha_{i,i+2}, \dots, \alpha_{i,2n-i+1}\}.$$

Note that we have the decomposition

$$\Phi^+_{B_n} = \mathcal{H}_1 \sqcup \cdots \sqcup \mathcal{H}_n.$$

We visualize $\Phi^+_{B_n}$ and its lower ideals by putting $\alpha_{i,j}$ in the jth row and ith column (so $\mathcal{H}_i$ appears in the ith column). See Figure 15 for this visualization of $\Phi^+_{B_3}$.

$\alpha_{1,2} = \mathbf{e}_1 - \mathbf{e}_2$		
$\alpha_{1,3} = \mathbf{e}_1 - \mathbf{e}_3$	$\alpha_{2,3} = \mathbf{e}_2 - \mathbf{e}_3$	
$\alpha_{1,4} = \mathbf{e}_1$	$\alpha_{2,4} = \mathbf{e}_2$	$\alpha_{3,4} = \mathbf{e}_3$
$\alpha_{1,5} = \mathbf{e}_1 + \mathbf{e}_3$	$\alpha_{2,5} = \mathbf{e}_2 + \mathbf{e}_3$	
$\alpha_{1,6} = \mathbf{e}_1 + \mathbf{e}_2$		

$\mathbf{e}_1 - \mathbf{e}_2$		
$\mathbf{e}_1 - \mathbf{e}_3$	$\mathbf{e}_2 - \mathbf{e}_3$	
	$\mathbf{e}_2$	$\mathbf{e}_3$
	$\mathbf{e}_2 + \mathbf{e}_3$	

FIGURE 15. Visualizations of $\Phi^+_{B_3}$ and its lower ideal $I = \{\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_1 - \mathbf{e}_3, \mathbf{e}_2 - \mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_3\}$.

For every lower ideal $I \subset \Phi^+_{B_n}$, we define the vector $h_I = (h_1, \ldots, h_n) \in \mathbb{Z}^n$ by

$$h_i = |I \cap \mathcal{H}_i| + i \quad \text{for } i = 1, 2, \ldots, n.$$

The partial order on $\Phi^+_{B_n}$ tells that I and h_I must satisfy the following properties:

(b1) $\alpha_{p,q} \in I$ implies $\alpha_{p,q-1} \in I$. In particular, $I \cap \mathcal{H}_i = \{\alpha_{i,i+1}, \ldots, \alpha_{i,h_i}\}$;
(b2) $\alpha_{p,q} \in I$ implies $\alpha_{p+1,q} \in I$.

We note that (b1) tells that the sequence h_I determines the lower ideal I. In particular, if $I \subset \Phi^+_{A_{n-1}}$, then h_I is nothing but the Hessenberg function associated with the ideal I (see Example 3.7).

Recall that for $\alpha = a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n \in V$, we write $\ell_\alpha = a_1x_1 + \cdots + a_nx_n \in \mathcal{R}$. Set $\ell_{i,j} = \ell_{\alpha_{i,j}}$. For $1 \le i \le j \le 2n+1-i$, we define

$$\psi^{B_n}_{i,j} = \sum_{k=1}^{i} \left(\prod_{l=i+1}^{j} \ell_{k,l} \right) \partial_k$$

where $\psi^{B_n}_{i,i} = \partial_1 + \cdots + \partial_i$. See Table 2 for the list of these elements for type B_3.

Theorem 8.1 (Abe-Horiguchi-Masuda-Murai-Sato)**.** *Let $I \subset \Phi^+_{B_n}$ be a lower ideal and $h_I = (h_1, \ldots, h_n)$. Then $\mathcal{D}(\mathcal{A}_I)$ is a free $\mathcal{R}$-module with basis $\{\psi^{B_n}_{i,h_i} \mid i = 1, 2, \ldots, n\}$.*

Remark 8.2. Since $\deg \psi^{B_n}_{i,j} = j - i$, the theorem tells that the sequence $(h_1 - 1, h_2 - 2, \ldots, h_n - n)$ is the exponents of $\mathcal{A}_I$. Indeed, it is not hard to see that this sequence coincides with the ideal exponents $d(I) = (d^I_1, \ldots, d^I_n)$ given in §4.3 up to permutations of entries.

We give a quick example before proving the theorem.

$\psi^{B_3}_{1,1}$	∂_1
$\psi^{B_3}_{2,2}$	$\partial_1+\partial_2$
$\psi^{B_3}_{3,3}$	$\partial_1+\partial_2+\partial_3$
$\psi^{B_3}_{1,2}$	$(x_1-x_2)\partial_1$
$\psi^{B_3}_{1,3}$	$(x_1-x_2)(x_1-x_3)\partial_1,$
$\psi^{B_3}_{1,4}$	$(x_1-x_2)(x_1-x_3)x_1\partial_1$
$\psi^{B_3}_{1,5}$	$(x_1-x_2)(x_1-x_3)x_1(x_1+x_3)\partial_1$
$\psi^{B_3}_{1,6}$	$(x_1-x_2)(x_1-x_3)x_1(x_1+x_3)(x_1+x_2)\partial_1$
$\psi^{B_3}_{2,3}$	$(x_1-x_3)\partial_1+(x_2-x_3)\partial_2$
$\psi^{B_3}_{2,4}$	$(x_1-x_3)x_1\partial_1+(x_2-x_3)x_2\partial_2$
$\psi^{B_3}_{2,5}$	$(x_1-x_3)x_1(x_1+x_3)\partial_1+(x_2-x_3)x_2(x_2+x_3)\partial_2$
$\psi^{B_3}_{3,4}$	$x_1\partial_1+x_2\partial_2+x_3\partial_3$

TABLE 2. List of elements $\psi^{B_3}_{i,j}$.

Example 8.3. Consider the lower ideal

$$I=\{\mathbf{e}_1-\mathbf{e}_2,\mathbf{e}_1-\mathbf{e}_3,\mathbf{e}_2-\mathbf{e}_3,\mathbf{e}_2,\mathbf{e}_2+\mathbf{e}_3,\mathbf{e}_3\}\subset\Phi^+_{B_3}.$$

(See Figure 15 for a visualization of this ideal.) Then $h_I=(3,5,4)$ and Theorem 8.1 tells that the following elements give a basis for $\mathcal{D}(\mathcal{A}_I)$:

$$\psi^{B_3}_{1,3}=(x_1-x_2)(x_1-x_3)\partial_1,$$
$$\psi^{B_3}_{2,5}=(x_1-x_3)x_1(x_1+x_3)\partial_1+(x_2-x_3)x_2(x_2+x_3)\partial_2,$$
$$\psi^{B_3}_{3,4}=x_1\partial_1+x_2\partial_2+x_3\partial_3.$$

Proof of Theorem 8.1. Each $\psi^{B_n}_{i,j}$ is an $\mathcal{R}$-linear combination of $\partial_1,\dots,\partial_i$, so it is clear that $\psi^{B_n}_{1,h_1},\dots,\psi^{B_n}_{n,h_n}$ are $\mathcal{R}$-independent. Also, since $\Phi^+_{B_n}=\mathcal{H}_1\bigsqcup\cdots\bigsqcup\mathcal{H}_n$ and $h_i-i=|\mathcal{H}_i\cap I|$ for all i, we have

$$\textstyle\sum_{i=1}^n\deg(\psi^{B_n}_{i,h_i})=\sum_{i=1}^n|I\cap\mathcal{H}_i|=|I|=|\mathcal{A}_I|.$$

Thus, by Saito's criterion, to prove the theorem it suffices to prove that each $\psi^{B_n}_{i,h_i}$ is contained in $\mathcal{D}(\mathcal{A}_I)$.

Fix $i\in[n]$ and $\alpha\in I$. We prove

$$\psi^{B_n}_{i,h_i}\cdot\ell_\alpha\in\ell_\alpha\mathcal{R}. \tag{8.1}$$

<u>**Case 1:**</u> Suppose $\alpha=\mathbf{e}_p\pm\mathbf{e}_q$ with $i<p<q$ or $\alpha=\mathbf{e}_p$ with $i<p$. Then, since ∂_p and ∂_q do not appear in $\psi^{B_n}_{i,h_i}$, we clearly have $\psi^{B_n}_{i,h_i}\cdot\ell_\alpha=0\in\ell_\alpha\mathcal{R}$ as desired.

<u>**Case 2:**</u> Suppose $\alpha=\mathbf{e}_p-\mathbf{e}_q$ with $p<q\le i$. Then

$$\psi^{B_n}_{i,h_i}\cdot\ell_\alpha=\left(\prod_{l=i+1}^{h_i}\ell_{p,l}\right)-\left(\prod_{l=i+1}^{h_i}\ell_{q,l}\right).$$

The above polynomial vanishes when we substitute $x_p = x_q$. This tells that $\psi^{B_n}_{i,h_i} \cdot \ell_\alpha \in \ell_\alpha \mathcal{R}$ as desired.

Case 3: Suppose $\alpha = \mathbf{e}_p + \mathbf{e}_q$ with $p < q \le i$. In this case, we must have $h_i = 2n - i + 1$. (Indeed, using (b1) and (b2), we have $\mathbf{e}_i + \mathbf{e}_{i+1} \in I$ when $i < n$ and $\mathbf{e}_n \in I$ when $i = n$.) Then

$$\psi^{B_n}_{i,h_i} \cdot \ell_\alpha = \left(\prod_{l=i+1}^{n} (x_p - x_l)(x_p + x_l) \right) x_p + \left(\prod_{l=i+1}^{n} (x_q - x_l)(x_q + x_l) \right) x_q.$$

The above polynomial vanish when we substitute $x_p = -x_q$, which tells $\psi^{B_n}_{i,h_i} \cdot \ell_\alpha \in \ell_\alpha \mathcal{R}$.

Case 4: Suppose $\alpha = \mathbf{e}_p \pm \mathbf{e}_q$ with $p \le i < q$ or $\alpha = \mathbf{e}_p$ with $p \le i$. Then we have $\alpha \in \mathcal{H}_p \cap I$ and

$$\psi^{B_n}_{i,h_i} \cdot \ell_\alpha = \prod_{l=i+1}^{h_i} \ell_{p,l} = \prod_{\ell \in \mathcal{H}_p \cap I} \ell \in \ell_\alpha \mathcal{R}$$

as desired.

Now Cases 1–4 prove the desired condition (8.1). ☐

For $1 \le i \le j \le 2n - i + 1$, let

$$f^{B_n}_{i,j} = \eta(\psi^{B_n}_{i,j}) = \textstyle\sum_{k=1}^{i} \left(\sum_{l=i+1}^{j} \ell_{k,l} \right) x_k.$$

By Theorem 7.11, we have the following presentation for $H^*(\mathrm{Hess}(\mathsf{N}, I))$ in type B_n.

Corollary 8.4. *With the same notation as in Theorem 8.1,*

$$H^*\big(\mathrm{Hess}(\mathsf{N}, I)\big) \cong \mathcal{R}/(f^{B_n}_{1,h_1}, \dots, f^{B_n}_{n,h_n}).$$

Example 8.5. Consider the ideal

$$I = \{\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_1 - \mathbf{e}_3, \mathbf{e}_2 - \mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_3\} \subset \Phi^+_{B_3}$$

in Example 8.3 (see also Figure 15). Then Corollary 8.4 and Table 2 tell

$$H^*\big(\mathrm{Hess}(\mathsf{N}, I)\big) \cong \mathcal{R}/(f_1, f_2, f_3)$$

with

$$\begin{aligned} f_1 &= f^{B_3}_{1,3} = (x_1 - x_2)(x_1 - x_3)x_1, \\ f_2 &= f^{B_3}_{2,5} = (x_1 - x_3)(x_1 + x_3)x_1^2 + (x_2 - x_3)(x_2 + x_3)x_2^2, \\ f_3 &= f^{B_3}_{3,4} = x_1^2 + x_2^2 + x_3^2. \end{aligned}$$

A special case of Corollary 8.4 when $I \subset \Phi^+_{A_{n-1}}$ (in this case $h_I \le (n, n, \dots, n)$) recovers the following result[14], which was originally proved by Abe-Harada-Horiguchi-Masuda [AHHM].

[14]In type A_{n-1}, we consider that Φ^+ is contained in the $(n-1)$-dimensional subspace $H_{(1,1,\dots,1)} \subset \mathbb{R}^n$, which is the kernel of the linear function $f_{n,n} = x_1 + \cdots + x_n$. One should apply Theorem 7.11 by setting $V = H_{(1,\dots,1)}$ and $\mathrm{sym}(V^*)$ should be considered as $\mathcal{R}/(x_1 + \cdots + x_n)$ in this case.

Corollary 8.6 (Abe-Harada-Horiguchi-Masuda). *Let* $I \subset \Phi^+_{A_{n-1}}$ *be an ideal and* $h_I = (h_1, \dots, h_n)$. *Then*

$$H^*\big(\mathrm{Hess}(\mathsf{N}, I)\big) \cong \mathcal{R}/(f_{1,h_1}, \dots, f_{n,h_n})$$

where

$$f_{i,j} = \sum_{k=1}^{i} \left(\prod_{l=i+1}^{j} (x_k - x_j) \right) x_k.$$

Remark 8.7. The polynomial $f_{i,j}$ in Corollary 8.6 is the same as the polynomial $\check{f}_{j,i}$ which appears in recursive formula of presentation of cohomology rings of type A regular nilpotent Hessenberg varieties given in §7 in Chapter 9 (be careful that i and j are switched).

Remark 8.8. The elements $\psi^{B_n}_{1,2n}, \psi^{B_n}_{2,2n-1}, \dots, \psi^{B_n}_{n,n+1}$ give a basis of $\mathcal{D}(\mathcal{A}_{\Phi^+_{B_n}})$, but as one can see from Table 2 they do not agree with the basis given in Theorem 4.9.

8.2. **Type C_n.** We will give a basis of $\mathcal{D}(\mathcal{A}_I)$ for lower ideals in $\Phi^+_{C_n}$. The construction is essentially the same as the type B case, so we only write statements and omit proofs.

Recall that $\Phi^+_{C_n}$ is a set

$$\Phi^+_{C_n} = \{\mathbf{e}_i \pm \mathbf{e}_j \mid 1 \le i < j \le n\} \cup \{2\mathbf{e}_i \mid i = 1, 2, \dots, n\}.$$

For $1 \le i < j \le 2n - i + 1$, we define

$$\alpha_{i,j} = \begin{cases} \mathbf{e}_i - \mathbf{e}_j & (i < j \le n), \\ \mathbf{e}_i + \mathbf{e}_{2n+1-j} & (n+1 \le j \le 2n - i + 1), \end{cases}$$

and

$$\mathcal{H}_i = \{\alpha_{i,i+1}, \alpha_{i,i+2}, \dots, \alpha_{i,2n-i+1}\}.$$

In the same way as for type B_n case, we visualize $\Phi^+_{C_n}$ and its lower ideals by putting $\alpha_{i,j}$ at the (j, i)th coordinate in the plane. See Figure 16.

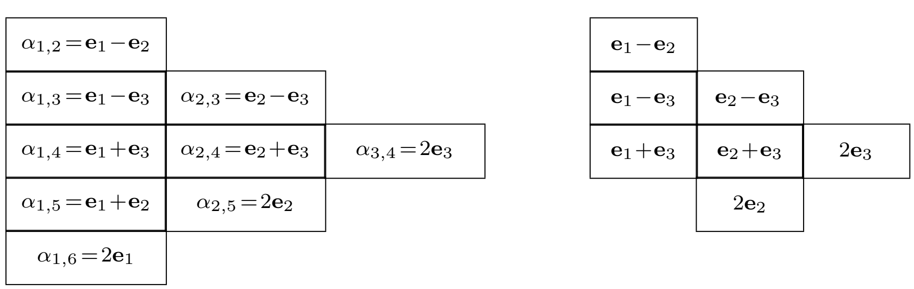


FIGURE 16. Visualizations of $\Phi^+_{C_3}$ and its ideal $I = \{\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_1 - \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_2 - \mathbf{e}_3, \mathbf{e}_2 + \mathbf{e}_3, 2\mathbf{e}_2, 2\mathbf{e}_3\}$.

For $1 \le i \le j \le 2n+1-i$, we define

$$\psi^{C_n}_{i,j} = \begin{cases} \sum_{k=1}^{i} \left(\prod_{l=i+1}^{j} \ell_{k,l}\right) \partial_k & \text{if } j \neq 2n+1-i, \\ \sum_{k=1}^{i} \left(\prod_{l=i+1}^{n} (x_k - x_l)(x_k + x_l)\right) x_k \partial_k & \text{if } j = 2n+1-i, \end{cases}$$

where $\psi^{C_n}_{i,i} = \partial_1 + \cdots + \partial_i$ and $\ell_{k,l} = \ell_{\alpha_{k,l}}$. We note that $\psi^{C_n}_{i,j} = \psi^{B_n}_{i,j}$ for $j \le n$ and $j = 2n+1-i$. Table 3 gives a list of elements $\psi^{C_3}_{i,j}$ (which are not equal to $\psi^{B_3}_{i,j}$).

$\psi^{C_3}_{1,4}$	$(x_1-x_2)(x_1-x_3)(x_1+x_3)\partial_1$
$\psi^{C_3}_{1,5}$	$(x_1-x_2)(x_1-x_3)(x_1+x_3)(x_1+x_2)\partial_1$
$\psi^{C_3}_{2,4}$	$(x_1-x_3)(x_1+x_3)\partial_1 + (x_2-x_3)(x_2+x_3)\partial_2$

Table 3. Elements $\psi^{C_3}_{i,j}$. Note that $\psi^{C_3}_{i,j} = \psi^{B_3}_{i,j}$ when $(i,j) \notin \{(1,4),(1,5),(2,4)\}$.

For a lower ideal $I \subset \Phi^+_{C_n}$, define the vector $h_I = (h_1, \ldots, h_n) \in \mathbb{Z}^n$ by

$$h_i = |I \cap \mathcal{H}_i| + i \quad \text{for } i = 1, 2, \ldots, n.$$

The following statements can be proved exactly in the same way as Theorem 8.1 and Corollary 8.4.

Theorem 8.9 (Abe-Horiguchi-Masuda-Murai-Sato)**.** *Let $I \subset \Phi^+_{C_n}$ be a lower ideal and $h_I = (h_1, \ldots, h_n)$. Then $\mathcal{D}(\mathcal{A}_I)$ is a free $\mathcal{R}$-module with basis $\{\psi^{C_n}_{i,h_i} \mid i = 1, 2, \ldots, n\}$.*

Corollary 8.10. *Let $f^{C_n}_{i,j} = \eta(\psi^{C_n}_{i,j})$ for all $1 \le i \le j \le 2n+1-i$. With the same notation as in Theorem 8.9, we have*

$$H^*\big(\mathrm{Hess}(\mathsf{N}, I)\big) \cong \mathcal{R}/(f^{C_n}_{1,h_1}, \ldots, f^{C_n}_{n,h_n}).$$

Example 8.11. Consider the lower ideal

$$I = \{\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_1 - \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_2 - \mathbf{e}_3, \mathbf{e}_2 + \mathbf{e}_3, 2\mathbf{e}_2, 2\mathbf{e}_3\} \subset \Phi^+_{C_3}.$$

(See Figure 16 for a visualization of this lower ideal.) Then $h_I = (4,5,4)$ and Theorem 8.9 tells that the following elements give a basis of $\mathcal{D}(\mathcal{A}_I)$:

$$\begin{aligned} \psi^{C_3}_{1,4} &= (x_1-x_2)(x_1-x_3)(x_1+x_3)\partial_1, \\ \psi^{C_3}_{2,5} &= (x_1-x_3)(x_1+x_3)x_1\partial_1 + (x_2-x_3)(x_2+x_3)x_2\partial_2, \\ \psi^{C_3}_{3,4} &= x_1\partial_1 + x_2\partial_2 + x_3\partial_3. \end{aligned}$$

In particular, we have

$$H^*\big(\mathrm{Hess}(\mathsf{N}, I)\big) \cong \mathbb{R}[x_1,x_2,x_3]/(f_1,f_2,f_3)$$

with

$$\begin{aligned} f_1 &= (x_1-x_2)(x_1-x_3)(x_1+x_3)x_1, \\ f_2 &= (x_1-x_3)(x_1+x_3)x_1^2 + (x_2-x_3)(x_2+x_3)x_2^2, \\ f_3 &= x_1^2+x_2^2+x_3^2. \end{aligned}$$

8.3. **Type D_n.** Finally we will give a basis of $\mathcal{D}(\mathcal{A}_I)$ for ideal arrangements of type D_n. The construction is a little complicated comparing types B_n and C_n. Recall that $\Phi^+_{D_n}$ is the set

$$\Phi^+_{D_n} = \{\mathbf{e}_i \pm \mathbf{e}_j \mid 1 \leq i < j \leq n\}.$$

For $1 \leq i \leq n-1$ and $i < j \leq 2n-i-1$, we set

$$\alpha_{i,j} = \begin{cases} \mathbf{e}_i - \mathbf{e}_j & (i < j \leq n), \\ \mathbf{e}_i + \mathbf{e}_{2n-j} & (n < j \leq 2n-i-1), \end{cases}$$

and

$$\gamma_k = \mathbf{e}_k + \mathbf{e}_n \quad \text{for } k = 1, 2, \ldots, n-1.$$

We consider the decomposition $\Phi^+_{D_n} = \mathcal{H}_1 \sqcup \mathcal{H}_2 \sqcup \cdots \sqcup \mathcal{H}_n$ defined by

$$\mathcal{H}_i = \{\alpha_{i,i+1}, \ldots, \alpha_{i,2n-i-1}\} \quad \text{for } i = 1, 2, \ldots, n-1$$

and

$$\mathcal{H}_n = \{\gamma_1, \ldots, \gamma_{n-1}\}.$$

To visualize $\Phi^+_{D_n}$ and its lower ideals, we place the root $\alpha_{i,j}$ at the (j,i)th position in the plane and place $\gamma_1, \ldots, \gamma_{n-1}$ in the independent row like Figure 17.

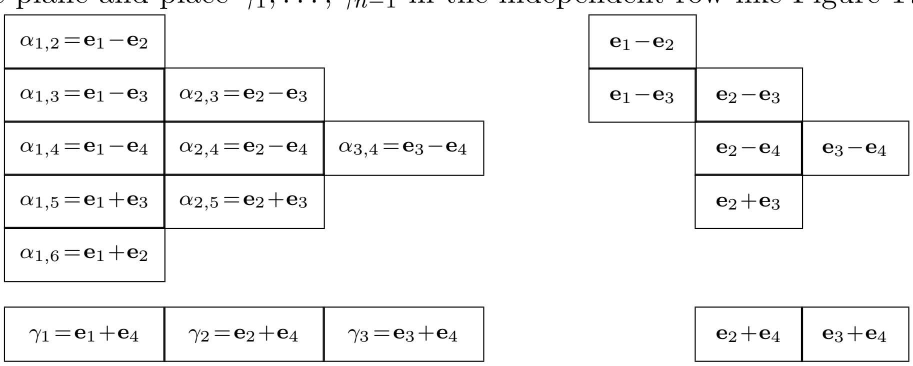


FIGURE 17. Visualizations of $\Phi^+_{D_4}$ and its lower ideal $I = \{\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_1 - \mathbf{e}_3, \mathbf{e}_2 - \mathbf{e}_3, \mathbf{e}_2 - \mathbf{e}_4, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_3 - \mathbf{e}_4, \mathbf{e}_3 + \mathbf{e}_4, \mathbf{e}_2 + \mathbf{e}_4\}$.

Let $\mathcal{Q} = Q(\mathcal{R})$ be the total fraction field of $\mathcal{R}$ and $\overline{\mathcal{R}} = \frac{1}{x_1 \cdots x_n}\mathcal{R} \subset \mathcal{Q}$ the $\mathcal{R}$-submodule of $\mathcal{Q}$ generated by $\frac{1}{x_1 \cdots x_n}$. To define elements that will give a basis of $\mathcal{D}(\mathcal{A}_I)$, we first define $\theta, g_{i,j} \in \mathrm{Der}(\mathcal{R}) \otimes_{\mathcal{R}} \overline{\mathcal{R}}$ by

$$g_{i,j} = \begin{cases} \sum_{k=1}^{i} \left(\prod_{l=i+1}^{j} \ell_{k,l} \right) \partial_k & (1 \leq i \leq n-1,\ i < j < n-1), \\ \sum_{k=1}^{i} \left(\prod_{l=i+1}^{j} \ell_{k,l} \right) \frac{x_k + x_n}{x_k} \partial_k & (1 \leq i \leq n-1,\ n-1 \leq j \leq 2n-i-1), \end{cases}$$

where $\ell_{k,l} = \ell_{\alpha_{k,l}}$, and

$$\theta = \sum_{k=1}^{n} \frac{1}{x_k} \partial_k.$$

Then we define

$$\psi^{D_n}_{i,j} = \begin{cases} g_{i,j} & (1 \leq i \leq n-1,\ i < j < n-1), \\ g_{i,j} + \left(\prod_{l=i+1}^{n} (-x_l) \right) \theta & (1 \leq i \leq n-1,\ j = n-1), \\ g_{i,j} - \left(\prod_{l=i+1}^{n} (-x_l) \right) \left(\prod_{m=2n-j}^{n} x_m \right) \theta & (1 \leq i \leq n-1,\ n \leq j \leq 2n-i-1), \end{cases}$$

$\psi_{1,1}^{D_4}$	∂_1
$\psi_{1,2}^{D_4}$	$(x_1-x_2)\partial_1$
$\psi_{1,3}^{D_4}$	$(x_1-x_2)(x_1-x_3)(x_1+x_4)\frac{1}{x_1}\partial_1 - x_2x_3x_4\theta$
$\psi_{1,4}^{D_4}$	$(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1+x_4)\frac{1}{x_1}\partial_1 + x_2x_3x_4^2\theta$
$\psi_{1,5}^{D_4}$	$(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1+x_4)(x_1+x_3)\frac{1}{x_1}\partial_1 + x_2x_3^2x_4^2\theta$
$\psi_{1,6}^{D_4}$	$(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1+x_4)(x_1+x_3)(x_1+x_2)\frac{1}{x_1}\partial_1 + x_2^2x_3^2x_4^2\theta$
$\psi_{2,2}^{D_4}$	$\partial_1+\partial_2$
$\psi_{2,3}^{D_4}$	$(x_1-x_3)(x_1+x_4)\frac{1}{x_1}\partial_1 + (x_2-x_3)(x_2+x_4)\frac{1}{x_2}\partial_2 + x_3x_4\theta$
$\psi_{2,4}^{D_4}$	$(x_1-x_3)(x_1-x_4)(x_1+x_4)\frac{1}{x_1}\partial_1 + (x_2-x_3)(x_2-x_4)(x_2+x_4)\frac{1}{x_2}\partial_2 - x_3x_4^2\theta$
$\psi_{2,5}^{D_4}$	$(x_1-x_3)(x_1-x_4)(x_1+x_4)(x_1+x_3)\frac{1}{x_1}\partial_1 + (x_2-x_3)(x_2-x_4)(x_2+x_4)(x_2+x_3)\frac{1}{x_2}\partial_2 - x_3^2x_4^2\theta$
$\psi_{3,3}^{D_4}$	$(x_1+x_4)\frac{1}{x_1}\partial_1 + (x_2+x_4)\frac{1}{x_2}\partial_2 + (x_3+x_4)\frac{1}{x_3}\partial_3 - x_4\theta \quad (= \partial_1+\partial_2+\partial_3-\partial_4)$
$\psi_{3,4}^{D_4}$	$(x_1-x_4)(x_1+x_4)\frac{1}{x_1}\partial_1 + (x_2-x_4)(x_2+x_4)\frac{1}{x_2}\partial_2 + (x_3-x_4)(x_3+x_4)\frac{1}{x_3}\partial_3 + x_4^2\theta$ $(= x_1\partial_1+x_2\partial_2+x_3\partial_3+x_4\partial_4)$
$\phi_4^{D_4}$	$(x_1-x_4)\frac{1}{x_1}\partial_1 + (x_2-x_4)\frac{1}{x_2}\partial_2 + (x_3-x_4)\frac{1}{x_3}\partial_3 + x_4\theta \quad (= \partial_1+\partial_2+\partial_3+\partial_4)$
$\phi_3^{D_4}$	$(x_1-x_3)(x_1-x_4)\frac{1}{x_1}\partial_1 + (x_2-x_3)(x_2-x_4)\frac{1}{x_2}\partial_2 - x_3x_4\theta$
$\phi_2^{D_4}$	$(x_1-x_2)(x_1-x_3)(x_1-x_4)\frac{1}{x_1}\partial_1 + x_2x_3x_4\theta$
$\phi_1^{D_4}$	$-x_1x_2x_3x_4\theta$

TABLE 4. Basis elements for type D_4. Note that $\theta = \frac{1}{x_1}\partial_1 + \cdots + \frac{1}{x_4}\partial_4$.

and

$$\phi_r^{D_n} = \big(\sum_{k=1}^{r-1}\big(\prod_{l=r}^{n}(x_k-x_l)\big)\tfrac{1}{x_k}\partial_k\big) - \big(\prod_{l=r}^{n}(-x_l)\big)\theta$$

for $r=1,2,\dots,n$. See Table 4 for the list of these elements for type D_4.

We note that $g_{i,j}$ and θ may not belong to $\mathrm{Der}(\mathcal{R})$ but $\psi_{i,j}^{D_n}$ does belong to $\mathrm{Der}(\mathcal{R})$. Indeed, for $j \geq n$, if we write $\psi_{i,j}^{D_n} = \frac{c_1}{x_1}\partial_1 + \cdots + \frac{c_n}{x_n}\partial_n$, then we have

$$c_k = \begin{cases} \left(\displaystyle\prod_{l=i+1}^{n}(x_k-x_l)\right)\left(\displaystyle\prod_{m=2n-j}^{n}(x_k+x_m)\right) - \left(\displaystyle\prod_{l=i+1}^{n}(-x_l)\right)\left(\displaystyle\prod_{m=2n-j}^{n}x_m\right) & \text{if } k \leq i, \\ \left(\displaystyle\prod_{l=i+1}^{n}(-x_l)\right)\left(\displaystyle\prod_{m=2n-j}^{n}x_m\right) & \text{if } k > i. \end{cases}$$

It is easy to see that each c_k is divisible by x_k so the coefficient of ∂_k in $\psi_{i,j}^{D_n}$ are polynomials. By similar computations, it is not hard to see that $\psi_{i,n-1}^{D_n}$ and $\phi_r^{D_n}$ are also elements of $\mathrm{Der}(\mathcal{R})$.

Let $I \subset \Phi_{D_n}^{+}$ be a lower ideal. We define the vector $h_I = (h_1,\dots,h_n) \in \mathbb{Z}^n$ by

$$h_i = |I \cap \mathcal{H}_i| + i \quad (i=1,2,\dots,n-1) \quad \text{and} \quad h_n = n - |I \cap \mathcal{H}_n|.$$

The partial order on $\Phi_{D_n}^{+}$ (see §3.3) tells that I must satisfy the following conditions

(d1) $\alpha_{p,q} \in I$ implies $\alpha_{p+1,q} \in I$ and $\alpha_{p,q-1} \in I$. In particular, one has $I \cap \mathcal{H}_i = \{\alpha_{i,i+1},\dots,\alpha_{i,h_i}\}$ for $i=1,2,\dots,n-1$.

(d2) $\gamma_p \in I$ implies $\gamma_{p+1} \in I$. In particular, $I \cap \mathcal{H}_n = \{\gamma_{h_n}, \gamma_{h_n+1}, \dots, \gamma_{n-1}\}$.
(d3) $\mathbf{e}_p + \mathbf{e}_q \in I$ implies $\mathbf{e}_p + \mathbf{e}_n \in I$ for $p < q \leq n$
(d4) $\gamma_p = \mathbf{e}_p + \mathbf{e}_n \in I$ with $p < n-1$ implies $\alpha_{p,n-1} = \mathbf{e}_p - \mathbf{e}_{n-1} \in I$.

Lemma 8.12. *Let $I \subset \Phi^+_{D_n}$ be a lower ideal and $h_I = (h_1, \dots, h_n)$. Then*
(1) *$\psi^{D_n}_{i,h_i} \in \mathcal{D}(\mathcal{A}_I)$ for $i = 1, 2, \dots, n-1$, and*
(2) *$\phi^{D_n}_{h_n} \in \mathcal{D}(\mathcal{A}_I)$.*

Proof. (1) Fix $1 \leq i \leq n-1$ and $\alpha \in I$. We prove

$$\psi^{D_n}_{i,h_i} \cdot \ell_\alpha \in \ell_\alpha \mathcal{R}. \tag{8.2}$$

Since $\theta \ell_\gamma \in \ell_\gamma \overline{\mathcal{R}}$ for any $\gamma \in \Phi^+_{D_n}$, to prove (8.2), it suffices to prove

$$g_{i,h_i} \cdot \ell_\alpha \in \ell_\alpha \overline{\mathcal{R}}.$$

__Case 1:__ Suppose $\alpha = \alpha_{p,q} \in \mathcal{H}_p$ with $p < n$. If $i < p$ then $g_{i,h_i} \cdot \ell_\alpha = 0 \in \ell_\alpha \overline{\mathcal{R}}$. Thus we may assume $i \geq p$.

__Subcase 1-1:__ Suppose $q \leq i$. Then $\alpha = \mathbf{e}_p - \mathbf{e}_q$ (as $q \leq n$) and $g_{i,h_i} \cdot \ell_\alpha$ equals to

$$\left(\prod_{l=i+1}^{h_i} \ell_{p,l}\right) - \left(\prod_{l=i+1}^{h_i} \ell_{q,l}\right) \text{ or } \left(\prod_{l=i+1}^{h_i} \ell_{p,l}\right) \frac{x_p + x_n}{x_p} - \left(\prod_{l=i+1}^{h_i} \ell_{q,l}\right) \frac{x_q + x_n}{x_q}.$$

Each of the above elements vanishes when we substitute $x_p = x_q$, so we have $g_{i,h_i} \cdot \ell_\alpha \in (x_p - x_q)\overline{\mathcal{R}} = \ell_\alpha \overline{\mathcal{R}}$.

__Subcase 1-2:__ Suppose $i < q \leq 2n - i - 1$. Then $\alpha = \mathbf{e}_p - \mathbf{e}_r$ or $\alpha = \mathbf{e}_p + \mathbf{e}_r$ with $r > i$, and by (d1) we have $\alpha - \mathbf{e}_p + \mathbf{e}_i \in I$. This tells that $\alpha \in \{\alpha_{p,i+1}, \dots, \alpha_{p,h_i}\}$, so we have $\ell_\alpha \in \{\ell_{p,i+1}, \dots, \ell_{p,h_i}\}$. Thus $g_{i,h_i} \cdot \ell_\alpha$ equals to

$$\left(\prod_{l=i+1}^{h_i} \ell_{p,l}\right) \quad \text{or} \quad \left(\prod_{\ell=i+1}^{h_i} \ell_{p,l}\right) \frac{x_p + x_n}{x_p}.$$

In both cases $g_{i,h_i} \cdot \ell_\alpha$ belongs to $\ell_\alpha \overline{\mathcal{R}}$, as desired.

__Subcase 1-3:__ Suppose $q > 2n - i - 1$. Then $\alpha = \mathbf{e}_p + \mathbf{e}_r$ with $r \leq i$. Also, by (d1) we have $h_i = 2n - i - 1$. Then we have

$$g_{i,h_i} \cdot \ell_\alpha = \left(\prod_{l=i+1}^{n} (x_p - x_l)(x_p + x_l)\right) \frac{1}{x_p} + \left(\prod_{l=i+1}^{n} (x_r - x_l)(x_r + x_l)\right) \frac{1}{x_r}.$$

This polynomial vanishes when we substitute $x_p = -x_r$, so $g_{i,h_i} \cdot \ell_\alpha \in (x_p + x_r)\overline{\mathcal{R}} = \ell_\alpha \overline{\mathcal{R}}$.

__Case 2:__ Suppose $\alpha = \mathbf{e}_p + \mathbf{e}_n$. If $i < p$ then $g_{i,h_i} \cdot \ell_\alpha = 0 \in \ell_\alpha \overline{\mathcal{R}}$. Assume $p \leq i < n$. Then we have $h_i \geq n-1$ since (d2) and (d4) tell $\mathbf{e}_i - \mathbf{e}_{n-1} \in I$ when $i < n-1$ (and $h_{n-1} \geq n-1$ by definition). This implies

$$g_{i,h_i} \cdot \ell_\alpha = \left(\prod_{l=i+1}^{h_i} \ell_{p,l}\right) \tfrac{x_p + x_n}{x_p} \in (x_p + x_n)\overline{\mathcal{R}} = \ell_\alpha \overline{\mathcal{R}}$$

as desired.

(2) Let $h_n = r$ and let

$$\phi = \sum_{k=1}^{r-1}\left(\prod_{l=r}^{n}(x_k - x_l)\right)\frac{1}{x_k}\partial_k.$$

Note that $\phi_{h_n}^{D_n} = \phi \pm (x_r \cdots x_n)\theta$. Fix $\alpha \in I$. We want to prove $\phi_{h_n}^{D_n} \cdot \ell_\alpha \in \ell_\alpha \mathcal{R}$. Since $\theta \cdot \ell_\alpha \in \ell_\alpha \overline{\mathcal{R}}$, it suffices to prove $\phi \cdot \ell_\alpha \in \ell_\alpha \overline{\mathcal{R}}$.

<u>Case 1:</u> Suppose $\alpha \in \mathcal{H}_p$ with $p < n$. If $r - 1 < p$ then $\phi \cdot \ell_\alpha = 0 \in \ell_\alpha \overline{\mathcal{R}}$, so we may assume $r - 1 \geq p$. Then since $h_n = r$ implies $\mathbf{e}_{r-1} + \mathbf{e}_n \notin I$, by (d2) and (d3) we have $\alpha = \mathbf{e}_p - \mathbf{e}_q$ for some $p < q$. If $p \leq r - 1 < q$ then

$$\phi \cdot \ell_\alpha = \phi \cdot (x_p - x_q) = (x_p - x_r) \cdots (x_p - x_n)\frac{1}{x_p} \in (x_p - x_q)\overline{\mathcal{R}}.$$

If $q \leq r - 1$, then one can see $\phi \cdot \ell_\alpha \in (x_p - x_q)\overline{\mathcal{R}}$ in the same way as in Subcase 1-1. In each case, we have $\phi \cdot \ell_\alpha \in \ell_\alpha \overline{\mathcal{R}}$ as desired.

<u>Case 2:</u> Suppose $\alpha \in \mathcal{H}_n$, that is, $\alpha = \mathbf{e}_p + \mathbf{e}_n$ for some $p < n$. Then $p \geq r$ by the definition of h_n and $\phi \cdot \ell_\alpha = 0 \in \ell_\alpha \overline{\mathcal{R}}$ as desired. ☐

Theorem 8.13 (Enokizono-Horiguchi-Nagaoka-Tsuchiya)**.** *Let $I \subset \Phi_{D_n}^+$ be a lower ideal and $h_I = (h_1, \ldots, h_n)$. Then $\mathcal{D}(\mathcal{A}_I)$ is a free $\mathcal{R}$-module with an $\mathcal{R}$-basis $\psi_{1,h_1}^{D_n}, \ldots, \psi_{n-1,h_{n-1}}^{D_n}, \phi_{h_n}^{D_n}$.*

Proof. We already proved that $\psi_{1,h_1}^{D_n}, \ldots, \psi_{n-1,h_{n-1}}^{D_n}, \phi_{h_n}^{D_n} \in \mathcal{D}(\mathcal{A}_I)$. Also, since $\deg \psi_{i,h_i}^{D_n} = |I \cap \mathcal{H}_i|$ for $i = 1, 2, \ldots, n-1$ and $\deg \phi_{h_n}^{D_n} = |I \cap \mathcal{H}_n|$, we have

$$\deg(\psi_{1,h_1}^{D_n}) + \cdots + \deg(\psi_{n-1,h_{n-1}}^{D_n}) + \deg(\phi_{h_n}^{D_n}) = |I| = |\mathcal{A}_I|.$$

Thus, by Saito's criterion, it suffices to prove that $\psi_{1,h_1}^{D_n}, \ldots, \psi_{n-1,h_{n-1}}^{D_n}, \phi_{h_n}^{D_n}$ are $\mathcal{R}$-independent, equivalently, the $\mathcal{Q}$-vector space

$$E = \operatorname{span}_{\mathcal{Q}}\{\psi_{1,h_1}^{D_n}, \ldots, \psi_{n-1,h_{n-1}}^{D_n}, \phi_{h_n}^{D_n}\}$$

equals to $U = \operatorname{span}_{\mathcal{Q}}\{\partial_1, \ldots, \partial_n\}$ (recall that $\mathcal{Q}$ is the field of fractions of $\mathcal{R}$). Observe that $\psi_{1,h_1}^{D_n}, \ldots, \psi_{n-1,h_{n-1}}^{D_n}$ generate the quotient space $U/(\mathcal{Q}\theta)$ since each $\psi_{i,h_i}^{D_n}$ is a linear combination of $\partial_1, \ldots, \partial_i$ modulo θ. Hence, to prove $E = U$, what we must prove is $\theta \in E$.

Let $h_n = r$. If $r = 1$, then $\phi_{h_n}^{D_n} = x_1 \cdots x_n \theta$, so $\theta \in E$ is clear. Assume $r > 1$. Then $\mathbf{e}_{r-1} + \mathbf{e}_n \notin I$ and $\mathbf{e}_r + \mathbf{e}_n \in I$ (unless $r = n$). The former condition together with (d2) and (d3) tell $\mathbf{e}_p + \mathbf{e}_q \notin I$ for all $p < q$ with $p \leq r - 1$, equivalently,

$$h_k \leq n \quad \text{for } k = 1, 2, \ldots, r-1. \tag{8.3}$$

(Indeed, if $\mathbf{e}_p + \mathbf{e}_q \in I$ for some $p < q$ with $p < r - 1$, then $\mathbf{e}_p + \mathbf{e}_n \in I$ by (d3) which implies $\mathbf{e}_{r-1} + \mathbf{e}_n \in I$ by (d2).) Let $\epsilon \leq r$ be the unique integer satisfying $h_1, \ldots, h_{\epsilon-1} \leq n - 2$ and $h_\epsilon \geq n - 1$. (We note that $h_r \geq n - 1$ since $\mathbf{e}_r + \mathbf{e}_n \in I$ if $r \neq n$.) The condition $h_1, \ldots, h_{\epsilon-1} \leq n - 2$ tells that $\psi_{1,h_1}^{D_n}, \ldots, \psi_{\epsilon-1,h_{\epsilon-1}}^{D_n}$ are $\mathcal{Q}$-linear combinations of $\partial_1, \ldots, \partial_{\epsilon-1}$, so we have

$$\partial_1, \ldots, \partial_{\epsilon-1} \in E.$$

Also, a straightforward computation shows

$$\phi_i^{D_n} + (x_i + x_n)\phi_{i+1}^{D_n} = \psi_{i,n}^{D_n} \quad \text{for } i = 1, 2, \ldots, n-1 \tag{8.4}$$

and

$$\psi_{i-1,n-1}^{D_n} + (x_i - x_n)\psi_{i,n-1}^{D_n} = \psi_{i,n}^{D_n} \quad \text{for } i = 2, \ldots, n-1. \tag{8.5}$$

Then, since $h_\epsilon, h_{\epsilon+1}, \ldots, h_{r-1} \in \{n-1, n\}$, by (8.5) we have $\psi_{k,n}^{D_n} \in E$ for $\epsilon + 1 \leq k < r$. But then (8.4) implies $\phi_{\epsilon+1}^{D_n} \in E$ since $\phi_r^{D_n} \in E$. Suppose $h_\epsilon = n$. Then (8.4) further implies $\phi_\epsilon^{D_n} \in E$. But $\phi_\epsilon^{D_n}$ equals to $-(\prod_{l=\epsilon}^n(-x_l))\theta$ modulo $\partial_1, \ldots, \partial_{\epsilon-1}$, so we have $\theta \in E$ as desired. Suppose $h_\epsilon = n-1$. Then

$$\phi_{\epsilon+1}^{D_n} \equiv (x_\epsilon - x_{\epsilon+1})\cdots(x_\epsilon - x_n)\tfrac{1}{x_\epsilon}\partial_\epsilon - (-x_{\epsilon+1})\cdots(-x_n)\theta \mod \partial_1, \ldots, \partial_{\epsilon-1}$$

and

$$\psi_{\epsilon,h_\epsilon}^{D_n} \equiv (x_\epsilon - x_{\epsilon+1})\cdots(x_\epsilon - x_{n-1})\tfrac{x_\epsilon + x_n}{x_\epsilon}\partial_\epsilon + (-x_{\epsilon+1})\cdots(-x_n)\theta \mod \partial_1, \ldots, \partial_{\epsilon-1}.$$

Now,

$$(x_\epsilon + x_n)\phi_{\epsilon+1}^{D_n} - (x_\epsilon - x_n)\psi_{\epsilon,h_{n-1}}^{D_n} \equiv (-x_{\epsilon+1})\cdots(-x_n)(-x_\epsilon - x_n - x_\epsilon + x_n)\cdot\theta$$

modulo $\partial_1, \ldots, \partial_{\epsilon-1}$. Since $\phi_{\epsilon+1}^{D_n}, \psi_{\epsilon,h_\epsilon}^{D_n} \in E$, we have $\theta \in E$ as desired. □

Corollary 8.14. *With the same notation as in Theorem 8.13, we have*

$$H^*\big(\mathrm{Hess}(\mathsf{N}, I)\big) \cong \mathcal{R}/(f_{1,h_1}, \ldots, f_{n,h_n}),$$

where $f_{k,h_k} = \eta(\psi_{k,h_k}^{D_n})$ *for* $k = 1, 2, \ldots, n-1$ *and* $f_{n,h_n} = \eta(\phi_{h_n}^{D_n})$.

Example 8.15. Consider the ideal

$$I = \{\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_1 - \mathbf{e}_3, \mathbf{e}_2 - \mathbf{e}_3, \mathbf{e}_2 - \mathbf{e}_4, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_3 - \mathbf{e}_4, \mathbf{e}_3 + \mathbf{e}_4, \mathbf{e}_2 + \mathbf{e}_4\} \subset \Phi_{D_4}^+.$$

(See Figure 17.) Then we have $h_I = (3, 5, 4, 2)$ and the module $\mathcal{D}(\mathcal{A}_I)$ is generated by $\psi_{1,3}^{D_4}, \psi_{2,5}^{D_4}, \psi_{3,4}^{D_4}, \phi_2^{D_4}$. Thus by Corollary 8.14 and Table 4 we have

$$H^*\big(\mathrm{Hess}(\mathsf{N}, I)\big) \cong \mathbb{R}[x_1, x_2, x_3, x_4]/(f_1, f_2, f_3, f_4)$$

with

$$\begin{aligned}
f_1 &= \eta(\psi_{1,3}^{D_4}) = (x_1 - x_2)(x_1 - x_3)(x_1 + x_4) - 4x_2x_3x_4, \\
f_2 &= \eta(\psi_{2,5}^{D_4}) = (x_1^2 - x_3^2)(x_1^2 - x_4^2) + (x_2^2 - x_3^2)(x_2^2 - x_4^2) - 4x_3^2x_4^2, \\
f_3 &= \eta(\psi_{3,4}^{D_4}) = x_1^2 + x_2^2 + x_3^2 + x_4^2, \\
f_4 &= \eta(\phi_2^{D_4}) = (x_1 - x_2)(x_1 - x_3)(x_1 - x_4) + 4x_2x_3x_4.
\end{aligned}$$

Note. The description of $\mathcal{D}(\mathcal{A}_I)$ for type B_n and C_n given in this section appeared in [AHMMS, §10], while the basis for type D_n was found by Enokizono-Horiguchi-Nagaoka-Tsuchiya [EHNT] (we note that our definition of h_I slightly differs from that in [EHNT].) A basis for exceptional types was also given in [EHNT].

Appendix: Properties of regular sequences

Let $\mathcal{R} = \mathbb{F}[x_1, \ldots, x_n]$ be a polynomial ring over a field $\mathbb{F}$. In this appendix, we briefly explain basic properties of regular sequences. Some other properties of regular sequences can be also found in Section 2 in Chapter 9.

In the rest of this section, $\mathbf{f} = [f_1, \ldots, f_m]$ always denotes a sequence of homogeneous polynomials in $\mathcal{R}$ of positive degrees. Recall that $\mathbf{f}$ is said to be a **regular sequence** for $\mathcal{R}$ if f_i is a non-zero divisor of $\mathcal{R}/(f_1, \ldots, f_{i-1})$ for $i = 1, 2, \ldots, m$. Let $\mathbf{f}' = [f_1, \ldots, f_{m-1}]$. If $\mathbf{f}$ is a regular sequence for $\mathcal{R}$, then we have a short exact sequence

$$0 \longrightarrow \mathcal{R}/(\mathbf{f}') \xrightarrow{\times f_m} \mathcal{R}/(\mathbf{f}') \longrightarrow \mathcal{R}/(\mathbf{f}) \longrightarrow 0. \tag{A.1}$$

Proposition A.1. *If $\mathbf{f} = [f_1, \ldots, f_m]$ is a regular sequence for $\mathcal{R}$, then*

$$\mathrm{Hilb}\big(\mathcal{R}/(\mathbf{f}), q\big) = \frac{1}{(1-q)^{n-m}} \prod_{k=1}^{m} (1 + q + \cdots + q^{\deg(f_k)-1}).$$

Proof. The short exact sequence (A.1) tells

$$\begin{aligned}\mathrm{Hilb}\big(\mathcal{R}/(\mathbf{f}), q\big) &= (1 - q^{\deg f_m})\mathrm{Hilb}(\mathcal{R}/(\mathbf{f}'), q)\\ &= (1-q)(1 + q + \cdots + q^{\deg(f_m)-1})\mathrm{Hilb}(\mathcal{R}/(\mathbf{f}'), q).\end{aligned}$$

Then the assertion follows from the fact that $\mathrm{Hilb}(\mathcal{R}, q) = \frac{1}{(1-q)^n}$ using induction on m. $\square$

To discuss more properties on regular sequences, we need Koszul complexes. Let $E_m = \bigwedge \mathbb{F}^m$ be the exterior algebra over $\mathbb{F}^m$. Let $\mathbf{e}_1, \ldots, \mathbf{e}_m$ be the standard basis of $\mathbb{F}^m$ and E_m^i the ith graded component of E_m. Thus if we write $\mathbf{e}_F = \mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_k}$ for $F = \{i_1, \ldots, i_k\} \subset [m]$ with $i_1 < \cdots < i_k$, then $\{\mathbf{e}_F \mid F \subset [m]\}$ is an $\mathbb{F}$-basis for E_m and $\{\mathbf{e}_F \mid F \subset [m],\ |F| = i\}$ is an $\mathbb{F}$-basis for E_m^i. For a sequence $\mathbf{f} = [f_1, \ldots, f_m]$ of elements in $\mathcal{R}$, the **Koszul complex** $K_\bullet(\mathbf{f})$ is the complex of $\mathcal{R}$-modules

$$0 \longrightarrow K_m(\mathbf{f}) \xrightarrow{d} K_{m-1}(\mathbf{f}) \xrightarrow{d} \cdots \longrightarrow K_1(\mathbf{f}) \xrightarrow{d} K_0(\mathbf{f}) \longrightarrow 0$$

such that $K_i(\mathbf{f}) = E_m^i \otimes_{\mathbb{F}} \mathcal{R}$ and the differential d is given by

$$d(\mathbf{e}_{j_1} \wedge \cdots \wedge \mathbf{e}_{j_i}) = \sum_{k=1}^{i} (-1)^{k-1} f_{j_k} (\mathbf{e}_{j_1} \wedge \cdots \wedge \hat{\mathbf{e}}_{j_k} \wedge \cdots \wedge \mathbf{e}_{j_i}).$$

We write $H_i(\mathbf{f}) = H_i(K_\bullet(\mathbf{f}))$. It is not hard to check the following properties.

Lemma A.2. *Let $\mathbf{f} = [f_1, \ldots, f_m]$ be a sequence of elements $\mathcal{R}$ and $\mathbf{f}' = [f_1, \ldots, f_{m-1}]$. Then*

1. $f \cdot H_k(\mathbf{f}) = 0$ *for any $f \in (f_1, \ldots, f_m)$ and any k.*
2. *One has the exact sequence*

$$\cdots \xrightarrow{\times(\pm f_m)} H_i(\mathbf{f}') \longrightarrow H_i(\mathbf{f}) \longrightarrow H_{i-1}(\mathbf{f}') \xrightarrow{\times(\pm f_m)} H_{i-1}(\mathbf{f}') \longrightarrow \cdots$$

Proof. To prove (1) we may assume $f = f_i$ for some i. The statement (1) follows from the fact that for any cycle z in $K_k(\mathbf{f})$ one has $d(\mathbf{e}_i \wedge z) = f_i z$. For the statement (2), it is straightforward that we have a short exact sequence of complexes

$$0 \longrightarrow K_\bullet(\mathbf{f}') \longrightarrow K_\bullet(\mathbf{f}) \longrightarrow K_\bullet(\mathbf{f})/K_\bullet(\mathbf{f}') \cong K_{\bullet-1}(\mathbf{f}') \longrightarrow 0$$

and the desired long exact sequence is obtained by taking the long exact sequence associated with the above short exact sequence. □

Proposition A.3. *A sequence* $\mathbf{f} = [f_1, \dots, f_m]$ *is a regular sequence for* $\mathcal{R}$ *if and only if* $K_\bullet(\mathbf{f})$ *is acyclic (that is,* $H_i(\mathbf{f}) = 0$ *for all* $i > 0$*).*

Proof. Let $\mathbf{f}' = [f_1, \dots, f_{m-1}]$. We prove the statement using induction on m.

Suppose that $\mathbf{f}$ is a regular sequence. Then $K_\bullet(\mathbf{f}')$ is acyclic by the induction hypothesis. Then by Lemma A.2(2) we have $H_i(\mathbf{f}) = 0$ for $i > 1$ and have an exact sequence

$$0 \longrightarrow H_1(\mathbf{f}) \longrightarrow H_0(\mathbf{f}') \stackrel{\times(\pm f_m)}{\longrightarrow} H_0(\mathbf{f}').$$

Since f_m is a non-zero divisor of $H_0(\mathbf{f}') = \mathcal{R}/(\mathbf{f}')$, the last map in the above sequence is injective, so we also have $H_1(\mathbf{f}) = 0$.

Suppose $K_\bullet(\mathbf{f})$ is acyclic. Then, again by Lemma A.2(2), we have the following short exact sequences

$$0 \longrightarrow H_0(\mathbf{f}') \stackrel{\times(\pm f_m)}{\longrightarrow} H_0(\mathbf{f}') \quad \text{and} \quad 0 \longrightarrow H_i(\mathbf{f}') \stackrel{\times(\pm f_m)}{\longrightarrow} H_i(\mathbf{f}') \longrightarrow 0 \ \ (\text{for } i \geq 1).$$

The former sequence tells that f_m is a non-zero divisor of $H_0(\mathbf{f}') = \mathcal{R}/(\mathbf{f}')$. The latter tells that the multiplication of f_m on $H_i(\mathbf{f}')$ is an isomorphism, which implies that $H_i(\mathbf{f}') = 0$ for all $i \geq 1$ (as f_m has a positive degree) and $\mathbf{f}'$ is a regular sequence for $\mathcal{R}$ by the induction hypothesis. Hence $\mathbf{f}$ is a regular sequence for $\mathcal{R}$. □

Proposition A.4. *If* $\mathcal{R}/(f_1, \dots, f_n)$ *is Artinian, then* $f_1, \dots, f_n$ *is a regular sequence for* $\mathcal{R}$*.*

Proof. Since $\mathcal{R}/(f_1, \dots, f_n)$ is Artinian, we have $x_1^p, \dots, x_n^p \in (f_1, \dots, f_n)$ for some $p > 0$. Let $\mathbf{f} = [f_1, \dots, f_n]$ and $\mathbf{y} = [x_1^p, \dots, x_n^p]$. Since $\mathbf{y}$ is a regular sequence, $H_i(\mathbf{y}) = 0$ for all $i > 0$ by Proposition A.3. Then, since Lemma A.2(2) tells that $H_i(g_1, \dots, g_k) = H_{i-1}(g_1, \dots, g_k) = 0$ implies $H_i(g_1, \dots, g_{k+1}) = 0$ for any sequence $g_1, \dots, g_{k+1} \in \mathcal{R}$, it follows that

$$H_i(\mathbf{f}, \mathbf{y}) \cong H_i(\mathbf{y}, \mathbf{f}) = 0 \quad \text{for } i > n. \tag{A.2}$$

On the other hand, for each $k = 1, 2, \dots, n$, since $x_k^p \in (\mathbf{f})$, by Lemma A.2(1) we have $x_k^p H_i(\mathbf{f}, x_1^p, \dots, x_{k-1}^p) = 0$ for all i. Then, by applying Lemma A.2(2) repeatedly we have

$$\text{if } H_j(\mathbf{f}) \neq 0 \text{ then } H_{j+k}(\mathbf{f}, x_1^p, \dots, x_{k-1}^p) \neq 0 \quad \text{for } k = 1, 2, \dots, n.$$

This, in particular, tells that

$$\text{if } H_j(\mathbf{f}) \neq 0 \text{ then } H_{j+n}(\mathbf{f}, \mathbf{y}) \neq 0. \tag{A.3}$$

Then (A.2) and (A.3) guarantee $H_i(\mathbf{f}) = 0$ for all $i > 0$ and therefore $\mathbf{f}$ is a regular sequence for $\mathcal{R}$. □

Recall that an $\mathbb{F}$-algebra $A = A_0 \oplus \cdots \oplus A_s$ is said to be a **Poincaré duality algebra** of socle degree s if $A_s \cong \mathbb{F}$ and the map

$$A_k \times A_{s-k} \to A_s, \quad (f,g) \mapsto fg \tag{A.4}$$

is non-degenerated for all k. For a graded $\mathbb{F}$-algebra $A = \mathcal{R}/I$, we write $\mathrm{Socle}(A) = \{f \in A \mid x_i f = 0 \text{ for all } i\}$.

Lemma A.5. *An Artinian graded $\mathbb{F}$-algebra $A = \mathcal{R}/I$ is a Poincaré duality algebra if and only if* $\mathrm{Socle}(A) \cong \mathbb{F}$.

Proof. Let $A = A_0 \oplus \cdots \oplus A_s$ with $A_s \neq 0$. Note that $\mathrm{Socle}(A) \supset A_s$. The only if part is clear, so we prove the if part. Suppose that $\mathrm{Socle}(A) \cong \mathbb{F}$. Then $\mathrm{Socle}(A)$ must be equal to A_s and $A_s \cong \mathbb{F}$. Let $f \in A_k$ with $k < s$. Since $\mathrm{Socle}(A) = A_s$, there is a linear form ℓ such that ℓf is non-zero in A. Applying this repeatedly, there is $g \in A_{s-k}$ such that $fg \in A_s$ is non-zero in A. This tells that the map (A.4) is non-degenerated. □

Proposition A.6. *Let $\mathbf{f} = [f_1, \dots, f_n]$ be a regular sequence for $\mathcal{R}$, $A = \mathcal{R}/(\mathbf{f})$, and let $g_{ij} \in \mathcal{R}$ be polynomials satisfying $f_j = \sum_{i=1}^n g_{ij} x_i$ for $j = 1, 2, \dots, n$. Then*

(1) *A is a Poincaré duality algebra of socle degree $s = \sum_{k=1}^n (\deg f_k - 1)$.*

(2) *$\det(g_{ij})$ is an $\mathbb{R}$-basis of A_s.*

Proof. Let $\boldsymbol{x} = [x_1, \dots, x_n]$. Consider the double complex $K_\bullet(\boldsymbol{x}) \otimes K_\bullet(\mathbf{f}) \cong K_\bullet(\boldsymbol{x}, \mathbf{f})$. Since $\mathbf{f}$ and $\boldsymbol{x}$ are regular sequences for $\mathcal{R}$, $K_\bullet(\mathbf{f})$ is a free resolution of $\mathcal{R}/(\mathbf{f})$ and $K_\bullet(\boldsymbol{x})$ is a free resolution of $\mathcal{R}/(\boldsymbol{x})$ by Proposition A.3. Recall by a standard result on homological algebra we have natural isomorphisms between the following three modules

$$H_i\big(K_\bullet(\boldsymbol{x}) \otimes \mathcal{R}/(\mathbf{f})\big) \xleftarrow{\cong} H_i\big(K_\bullet(\boldsymbol{x}) \otimes K_\bullet(\mathbf{f})\big) \xrightarrow{\cong} H_i\big(\mathcal{R}/(\boldsymbol{x}) \otimes K_\bullet(\mathbf{f})\big), \tag{A.5}$$

and $\mathrm{Tor}_i(\mathcal{R}/(\boldsymbol{x}), \mathcal{R}/(\boldsymbol{f}))$ is defined to be one of the above isomorphic modules. See [Rot, Theorem 10.22].

(1) Proposition A.1 tells $A = A_0 \oplus \cdots \oplus A_s$ with $A_s \cong \mathbb{F}$, so by Lemma A.5 it suffices to prove that $\mathrm{Socle}(A) \cong \mathbb{F}$. By the definition of Koszul complexes we have

$$\mathrm{Socle}\big(\mathcal{R}/(\mathbf{f})\big) \cong H_n\big(K_\bullet(\boldsymbol{x}) \otimes_{\mathcal{R}} \mathcal{R}/(\mathbf{f})\big).$$

Then by (A.5) we have

$$H_n\big(K_\bullet(\boldsymbol{x}) \otimes \mathcal{R}/(\mathbf{f})\big) \cong H_n\big(K_\bullet(\mathbf{f}) \otimes \mathcal{R}/(\boldsymbol{x})\big) \cong \mathcal{R}/(\boldsymbol{x}) \cong \mathbb{F},$$

which proves $\mathrm{Socle}(\mathcal{R}/(\mathbf{f})) \cong \mathbb{F}$ as desired. (We use $H_i(K_\bullet(\mathbf{f}) \otimes \mathcal{R}/(\boldsymbol{x})) = K_i(\mathbf{f}) \otimes \mathcal{R}/(\boldsymbol{x})$ for the third isomorphism.)

(2) What we must prove is that $\det(g_{ij})$ is a non-zero element of $\mathrm{Socle}(A)$. Let $\mathbf{e}_1, \dots, \mathbf{e}_n, \mathbf{e}'_1, \dots, \mathbf{e}'_n$ be the basis of $K_1(\boldsymbol{x}, \mathbf{f})$ with $d(\mathbf{e}_i) = x_i$ and $d(\mathbf{e}'_i) = f_i$ for all i. Let $u_j = \sum_{i=1}^n g_{ij} \mathbf{e}_i$ for $j = 1, 2, \dots, n$. Then $d(u_k) = f_k = d(\mathbf{e}'_k)$ for each k, so $u_k - \mathbf{e}'_k$ is a cycle of $K_\bullet(\boldsymbol{x}) \otimes K_\bullet(\mathbf{f})$. Set $z = (u_1 - \mathbf{e}'_1) \wedge \cdots \wedge (u_n - \mathbf{e}'_n)$. Since products of cycles are cycles in Koszul complexes, z is a cycle of $K_\bullet(\boldsymbol{x}) \otimes K_\bullet(\mathbf{f})$. Also, by the isomorphisms (A.5), the image of z in $H_n(\mathcal{R}/(\boldsymbol{x}) \otimes K_\bullet(\mathbf{f}))$ is $\pm \mathbf{e}'_1 \wedge \cdots \wedge \mathbf{e}'_n$ and the image of z in $H_n(K_\bullet(\boldsymbol{x}) \otimes \mathcal{R}/(\mathbf{f}))$ is $u_1 \wedge \cdots \wedge u_n$. Since $\mathbf{e}'_1 \wedge \cdots \wedge \mathbf{e}'_n \neq 0$ in $H_n(\mathcal{R}/(\boldsymbol{x}) \otimes K_\bullet(\mathbf{f})) = K_n(\mathbf{f}) \otimes (\mathcal{R}/(\boldsymbol{x}))$, we have $u_1 \wedge \cdots \wedge u_n \neq 0$. Since

$u_1\wedge\cdots\wedge u_n = \det(g_{ij})\mathbf{e}_1\wedge\cdots\wedge\mathbf{e}_n$ and $H_n(K_\bullet(\boldsymbol{x})\otimes\mathcal{R}/(\mathbf{f})) = \mathrm{Socle}(\mathcal{R}/(\mathbf{f}))\mathbf{e}_1\wedge\cdots\wedge\mathbf{e}_n$, it follows that $\det(g_{ij})$ is a non-zero element in $\mathrm{Socle}(A)$ as desired. □

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