

h -VECTORS OF SIMPLICIAL CELL BALLS

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ABSTRACT. A simplicial cell ball is a simplicial poset whose geometric realization is homeomorphic to a ball. Recently, Samuel Kolins gave a series of necessary conditions and sufficient conditions on h -vectors of simplicial cell balls, and characterized them up to dimension 6. In this paper, we extend Kolins' results. We characterize all possible h -vectors of simplicial cell balls in arbitrary dimension.

1. INTRODUCTION

A *simplicial poset* is a finite poset P with a minimal element $\hat{0}$ such that every interval $[\hat{0}, \sigma]$ for $\sigma \in P$ is a Boolean algebra. A typical example of a simplicial poset is the face poset of a simplicial complex, but not all simplicial posets come from simplicial complexes. Simplicial posets are CW-posets. Thus for any simplicial poset P there is a regular CW-complex $\Gamma(P)$ whose face poset is isomorphic to P (see [Bj1, pp. 8–9]). A simplicial poset P (or a CW-complex $\Gamma(P)$) is called a *simplicial cell d -ball* (respectively *d -sphere*) if $\Gamma(P)$ is homeomorphic to a d -ball (respectively d -sphere). Kolins [Ko] studied h -vectors of simplicial cell balls and gave a characterization of h -vectors of simplicial cell balls up to dimension 6. In this paper, we extend the results of Kolins and give a complete characterization of h -vectors of simplicial cell balls.

Let P be a simplicial poset. We say that an element $\sigma \in P$ has *rank i* , denoted $\text{rank } \sigma = i$, if $[\hat{0}, \sigma]$ is a Boolean algebra of rank i . The *dimension* of P is

$$\dim P = \max\{\text{rank } \sigma : \sigma \in P\} - 1.$$

Let $d = \dim P + 1$ and let $f_i = f_i(P)$ be the number of elements $\sigma \in P$ having rank $i + 1$ for $i = -1, 0, \dots, d - 1$. Thus f_i is the number of i -dimensional cells of $\Gamma(P)$. The vector $f(P) = (f_{-1}, f_0, \dots, f_{d-1})$ is called the *f -vector* (face vector) of P . We define the *h -vector* $h(P) = (h_0(P), h_1(P), \dots, h_d(P)) \in \mathbb{Z}^{d+1}$ of P by the relation

$$\sum_{i=0}^d f_{i-1}(1-t)^{d-i} = \sum_{i=0}^d h_i(P)t^{d-i}.$$

Then knowing $f(P)$ is equivalent to knowing $h(P)$.

On face vectors of simplicial cell spheres, Stanley [St1] and Masuda [Ma] proved the following result, which characterizes all possible h -vectors of simplicial cell spheres.

Theorem 1.1 (Stanley, Masuda). *Let $\mathbf{h} = (h_0, h_1, \dots, h_d) \in \mathbb{Z}^{d+1}$. Then \mathbf{h} is the h -vector of a simplicial cell $(d - 1)$ -sphere if and only if it satisfies the following conditions:*

- (1) $h_0 = h_d = 1$ and $h_i = h_{d-i}$ for $i = 1, 2, \dots, d-1$.
- (2) $h_i \geq 0$ for $i = 0, 1, \dots, d$.
- (3) If $h_n = 0$ for some $1 \leq n \leq d-1$ then $\sum_{k=0}^d h_k$ is even.

For a vector $\mathbf{h} = (h_0, h_1, \dots, h_d) \in \mathbb{Z}^{d+1}$, we define $\partial\mathbf{h} = (\partial h_0, \partial h_1, \dots, \partial h_{d-1}) \in \mathbb{Z}^d$ by

$$\partial h_i = (h_0 + h_1 + \dots + h_i) - (h_d + h_{d-1} + \dots + h_{d-i})$$

for $i = 0, 1, \dots, d-1$. It is known that if \mathbf{h} is the h -vector of a simplicial cell $(d-1)$ -ball P , then $\partial\mathbf{h}$ is the h -vector of the boundary sphere of P . In this paper, we prove the next result, which characterizes all possible h -vectors of simplicial cell balls.

Theorem 1.2. *Let $\mathbf{h} = (h_0, h_1, \dots, h_d) \in \mathbb{Z}^{d+1}$. Then \mathbf{h} is the h -vector of a simplicial cell $(d-1)$ -ball if and only if it satisfies the following conditions:*

- (1) $h_0 = 1$, $h_d = 0$ and $h_k \geq 0$ for $k = 1, 2, \dots, d-1$.
- (2) $\partial h_k \geq 0$ for $k = 0, 1, \dots, d-1$.
- (3) If d is odd and $\partial h_n = 0$ for some $1 \leq n \leq d-2$ then $\sum_{k=0}^d h_k$ is even.
- (4) If $\partial h_n = 0$ for some $1 \leq n \leq d-2$ then

$$h_k + h_{k-1} + \dots + h_{k-n+1} \geq \partial h_k \quad \text{for } k = n, n+1, \dots, d-1.$$

- (5) If $\partial h_i = 0$ and $h_j = 0$ for some positive integers i and j with $i+j \leq d$ then $\sum_{k=0}^d h_k$ is even.
- (6) Suppose $\partial h_n = 0$ for some $1 \leq n < \frac{d}{2}$. If $(h_\ell + h_{\ell-1} + \dots + h_{\ell-n+1}) - \partial h_\ell < n$ for some $n \leq \ell \leq d-n$ then $\sum_{k=0}^d h_k$ is even.
- (7) Suppose $\partial h_i = 0$ and $h_j = 0$ for some integers i and j with $0 < i < \frac{d}{2}$ and $d-i < j < d$. If $\partial h_\ell \leq \ell$ for some $\ell \leq d-j$ then $\sum_{k=0}^d h_k$ is even.

Note that conditions (5), (6) and (7) are unnecessary when d is odd. We also note that some special cases of the above theorem are due to Kolins [Ko]. For necessity, (1), (2), (3) and some special cases of (4), (5) and (7) (the condition (4) for $n=1$, the condition (5) for $i=1, 2$ and $j=1$, and the condition (7) for $\ell=1$) were proved in [Ko, Sections 3, 4 and 5]. Also, when $\partial\mathbf{h}$ is positive or $\sum_{k=0}^d h_k$ is even, the sufficiency of the theorem was proved in [Ko, Theorem 16]. In particular, for even dimensional simplicial cell balls, the sufficiency of the theorem is due to Kolins.

A simplicial poset P (or a CW-complex $\Gamma(P)$) is called a *simplicial cell decomposition of a topological space X* if $\Gamma(P)$ is homeomorphic to X . It was proved in [Mu] that face vectors of simplicial cell decompositions of some manifolds without boundary, such as product of spheres and real projective spaces, are characterized by conditions similar to those in Theorem 1.1. It would be of interest to find manifolds M with boundary such that face vectors of simplicial cell decompositions of M can be characterized by conditions similar to those in Theorem 1.2.

This paper is organized as follows. In Section 2, we prove the necessity of Theorem 1.2 by using the face ring of a simplicial poset. In Section 3, we prove the sufficiency of Theorem 1.2 by using constructibility and shellability.

2. PROOF OF NECESSITY

In this section, we prove the necessity of Theorem 1.2. To prove this, we need the face ring of a simplicial poset introduced by Stanley [St1]. We assume familiarity with commutative algebra theory.

We recall some basic notations on commutative algebra. Let K be an infinite field, $R = R_0 \oplus R_1 \oplus \cdots$ a finitely generated commutative graded K -algebra with $R_0 = K$, where R_k is the graded component of R of degree k , and let M be a finitely generated graded R -module. The *Hilbert series* $H_M(t)$ of M is the formal power series $H_M(t) = \sum_{k=0}^{\infty} (\dim_K M_k) t^k$. The *Krull dimension* of M , denoted $\dim M$, is the minimal m for which there exist homogeneous elements $\theta_1, \dots, \theta_m \in R$ of positive degrees such that $M/(\theta_1, \dots, \theta_m)M$ is a finite dimensional K -vector space. When $\dim M = d$, a sequence of homogeneous elements $\theta_1, \dots, \theta_d \in R$ of positive degrees such that $M/(\theta_1, \dots, \theta_d)M$ is a finite dimensional K -vector space is called a *homogeneous system of parameters* (*h.s.o.p.* for short) of M . Moreover, if $\theta_1, \dots, \theta_d$ have degree 1, then we call $\theta_1, \dots, \theta_d$ a *linear system of parameters* (*l.s.o.p.* for short) of M . We say that M is *Cohen-Macaulay* if, for every (equivalently some) h.s.o.p. $\theta_1, \dots, \theta_d$ of M , one has that θ_i is a non-zero divisor of $M/(\theta_1, \dots, \theta_{i-1})M$ for all $i = 1, 2, \dots, d$. Also, R is said to be *Cohen-Macaulay* if it is a Cohen-Macaulay R -module.

Let P be a $(d - 1)$ -dimensional simplicial poset and let $A = K[x_\sigma : \sigma \in P \setminus \{\hat{0}\}]$ be the polynomial ring over an infinite field K in indeterminates indexed by the elements in $P \setminus \{\hat{0}\}$. We define the grading of A by $\deg x_\sigma = \text{rank } \sigma$. The *face ring of P* is the quotient ring $K[P] = A/I_P$, where I_P is the ideal generated by the following elements:

- $x_\sigma x_\tau$, if $\sigma, \tau \in P$ have no common upper bounds in P .
- $x_\sigma x_\tau - x_{\sigma \wedge \tau} \sum_{\rho} x_\rho$, where the summation runs over the all minimal upper bounds of σ and τ and where $\sigma \wedge \tau$ is the meet (largest lower bound) of σ and τ , otherwise. (We consider $x_{\sigma \wedge \tau} = 1$ if $\sigma \wedge \tau = \hat{0}$.)

Note that, for all $\sigma, \tau \in P$ such that σ and τ have a common upper bound ρ , there is the unique largest lower bound of σ and τ since $[\hat{0}, \rho]$ is a Boolean algebra. Also, since generators of I_P are homogeneous, the ring $K[P]$ is graded. In the special case when P is the face poset of a simplicial complex, the ring $K[P]$ is isomorphic to the face ring (Stanley-Reisner ring) of a simplicial complex [St2, p. 53]. See [MMP, p. 467].

A simplicial poset P is said to be *Cohen-Macaulay* if the ring $K[P]$ is Cohen-Macaulay. Note that this is equivalent to saying that the order complex of $P \setminus \{\hat{0}\}$ is a Cohen-Macaulay simplicial complex. On face rings of simplicial posets, the following properties are known.

- The Krull dimension of $K[P]$ is $\dim P + 1$.
- $K[P]$ has an l.s.o.p.
- $H_{K[P]}(t) = (h_0 + h_1 t + \cdots + h_d t^d)/(1 - t)^d$, where $d = \dim P + 1$ and where $(h_0, h_1, \dots, h_d) = h(P)$.
- If $\Gamma(P)$ is homeomorphic to a ball or a sphere then P is Cohen-Macaulay.

See [St1, pp. 325–326] for the first three properties. The last property follows from a topological criterion of Cohen-Macaulay simplicial complexes [St2, II, Corollary 4.2 and Proposition 4.3].

A subset I of a simplicial poset P is said to be an *order ideal* of P if $\sigma \in I$ and $\tau \leq \sigma$ imply $\tau \in I$. Thus an order ideal of a simplicial poset is again a simplicial poset. For a simplicial poset P and elements $\sigma_1, \dots, \sigma_k \in P$, we write $\langle \sigma_1, \dots, \sigma_k \rangle$ for the order ideal of P generated by $\sigma_1, \dots, \sigma_k$, in other words,

$$\langle \sigma_1, \dots, \sigma_k \rangle = \{ \tau \in P : \text{there is } 1 \leq j \leq k \text{ such that } \tau \leq \sigma_j \}.$$

For elements f_1, \dots, f_m in a ring A , we write (f_1, \dots, f_m) for the ideal of A generated by f_1, \dots, f_m . We often use the following obvious fact.

Lemma 2.1. *Let P be a simplicial poset and $A = K[x_\sigma : \sigma \in P \setminus \{\hat{0}\}]$. If Q is an order ideal of P then $A/(I_P + (x_\sigma : \sigma \notin Q))$ is isomorphic to $K[Q]$ as a ring.*

Let P be a simplicial cell $(d-1)$ -ball. Then each rank $d-1$ element of P is covered by at most two elements. The *boundary* ∂P of P is the order ideal

$$\partial P = \langle \sigma \in P : \text{rank } \sigma = d-1, \sigma \text{ is covered by exactly one element in } P \rangle.$$

Since ∂P is the face poset of the boundary cell complex of $\Gamma(P)$, ∂P is a simplicial cell $(d-2)$ -sphere. Moreover, it is known that if $\mathbf{h} = (h_0, h_1, \dots, h_d)$ is the h -vector of P then $\partial \mathbf{h} = (\partial h_0, \partial h_1, \dots, \partial h_{d-1})$ is the h -vector of ∂P . See [Ko, Section 3]. On h -vectors of simplicial cell balls, the following result was proved in [Ko, Theorem 5].

Theorem 2.2 (Kolins). *Let P be a simplicial cell $(d-1)$ -ball and $\mathbf{h} = h(P) = (h_0, h_1, \dots, h_d)$. Then*

- (1) $h_0 = 1, h_d = 0$ and $h_k \geq 0$ for $k = 1, 2, \dots, d-1$.
- (2) $\partial h_k \geq 0$ for $k = 0, 1, \dots, d-1$.
- (3) If d is odd and $\partial h_n = 0$ for some $1 \leq n \leq d-2$ then $\sum_{k=0}^d h_k$ is even.

We sketch a proof of the above theorem. The vectors \mathbf{h} and $\partial \mathbf{h}$ are non-negative since h -vectors of Cohen-Macaulay simplicial posets are non-negative [St1, Theorem 3.10]. Also, since $h_d(P)$ is equal to the reduced Euler characteristic of P times $(-1)^{d-1}$, $h_d(P) = 0$ if P is a simplicial cell ball. Finally, (3) follows from Theorem 1.1 as follows: If d is odd then

$$\partial h_{\frac{d-1}{2}} = (h_0 + \dots + h_{\frac{d-1}{2}}) - (h_{\frac{d+1}{2}} + \dots + h_d)$$

is equal to $\sum_{k=0}^d h_k \pmod{2}$. Since $\sum_{k=0}^{d-1} \partial h_k$ is even by Theorem 1.1(3), by the symmetry $\partial h_i = \partial h_{d-1-i}$ of $\partial \mathbf{h}$, $\partial h_{\frac{d-1}{2}}$ must be even.

In the rest of this section, we prove that conditions (4), (5), (6) and (7) in Theorem 1.2 are necessary conditions of h -vectors of simplicial cell balls.

2.1. Proof of (4).

Lemma 2.3. *Let P be a $(d-1)$ -dimensional simplicial poset and $h(P) = (h_0, \dots, h_d)$. Let $K[P] = A/I_P$ be the face ring of P and $\theta_1, \dots, \theta_d \in A_1$ an l.s.o.p. of $K[P]$. If $K[P]$ is Cohen-Macaulay then, for all integers $n > 0$ and $n \leq k \leq d$, one has*

$$\dim_K (A/(I_P + (\theta_1, \dots, \theta_{d-1}, \theta_d^n)))_k = h_k + h_{k-1} + \dots + h_{k-n+1}.$$

Proof. Let $R[i] = A/(I_P + (\theta_1, \dots, \theta_i))$ for $i = 0, 1, \dots, d-1$. Since $R[0] = K[P]$ is Cohen-Macaulay, we have the exact sequence

$$0 \longrightarrow R[i-1] \xrightarrow{\times \theta_i} R[i-1] \longrightarrow R[i] \longrightarrow 0$$

for $i = 1, 2, \dots, d-1$. The exact sequences show that $H_{R[i]}(t) = (1-t)H_{R[i-1]}(t)$ for $i = 1, 2, \dots, d-1$. Then, since $H_{K[P]}(t) = (h_0 + h_1t + \dots + h_d t^d)/(1-t)^d$, we have $H_{R[d-1]}(t) = (h_0 + h_1t + \dots + h_d t^d)/(1-t)$. This fact says

$$\dim_K (R[d-1])_k = h_k + h_{k-1} + \dots + h_1 + h_0$$

for $k = 0, 1, \dots, d$. Also, since $\theta_1, \dots, \theta_{d-1}, \theta_d^n$ is an h.s.o.p. of $K[P]$, the sequence

$$0 \longrightarrow R[d-1] \xrightarrow{\times \theta_d^n} R[d-1] \longrightarrow A/(I_P + (\theta_1, \dots, \theta_{d-1}, \theta_d^n)) \longrightarrow 0$$

is exact. Hence

$$\begin{aligned} \dim_K (A/(I_P + (\theta_1, \dots, \theta_{d-1}, \theta_d^n)))_k &= \dim_K (R[d-1])_k - \dim_K (R[d-1])_{k-n} \\ &= h_k + h_{k-1} + \dots + h_{k-n+1} \end{aligned}$$

for $k = n, n+1, \dots, d$, as desired. \square

Theorem 2.4. *Let P be a simplicial cell $(d-1)$ -ball and $\mathbf{h} = h(P) = (h_0, h_1, \dots, h_d)$. If $\partial h_n = 0$ for some $1 \leq n \leq d-2$ then*

$$h_k + h_{k-1} + \dots + h_{k-n+1} \geq \partial h_k \quad \text{for } k = n, n+1, \dots, d-1.$$

Proof. Let $A = K[x_\sigma : \sigma \in P \setminus \{\hat{0}\}]$ and $J = I_P + (x_\sigma : \sigma \notin \partial P) \subset A$. Then $A/J \cong K[\partial P]$ by Lemma 2.1. For a general choice of linear forms $\theta_1, \dots, \theta_d \in A_1$, we have (e.g. use a criterion of an l.s.o.p. for face rings [Ko, Lemma 6])

- $\theta_1, \dots, \theta_{d-1}$ is an l.s.o.p. of A/J .
- $\theta_1, \dots, \theta_{d-1}, \theta_d$ is an l.s.o.p. of A/I_P .

Let $\bar{\Theta} = \theta_1, \dots, \theta_{d-1}$. Since A/J is a $(d-1)$ -dimensional Cohen-Macaulay ring,

$$\dim_K (A/(J + (\bar{\Theta})))_k = h_k(\partial P) = \partial h_k$$

for $k = 0, 1, \dots, d-1$. In particular, since $\dim_K(A/(J + (\bar{\Theta})))_n = \partial h_n = 0$ by the assumption, we have $\theta_d^n \in J + (\bar{\Theta})$. Then, since $J \supset I_P$, we have $J + (\bar{\Theta}) \supset I_P + (\bar{\Theta}) + (\theta_d^n)$. Thus

$$\dim_K (A/(I_P + (\bar{\Theta}) + (\theta_d^n)))_k \geq \dim_K (A/(J + (\bar{\Theta})))_k = \partial h_k$$

for all $k = 0, 1, \dots, d-1$. Then the statement follows from Lemma 2.3. \square

2.2. Proof of (5), (6) and (7).

Let P be a $(d-1)$ -dimensional simplicial poset. A rank 1 element of P is called a *vertex* of P and a maximal element (w.r.t. a partial order) of P is called a *facet* of P . The *cone* of P is the product of the posets

$$C(P) = P \times \{1, 2\} = \{(\sigma, i) : \sigma \in P, i = 1, 2\},$$

where the ordering is defined by $(\sigma, i) \geq (\tau, j)$ if $\sigma \geq \tau$ and $i \geq j$. Then $C(P)$ is simplicial and $\Gamma(C(P))$ is homeomorphic to the (topological) cone of $\Gamma(P)$. Also,

a straightforward computation shows $h_i(P) = h_i(C(P))$ for all $i = 0, 1, \dots, d$ and $h_{d+1}(C(P)) = 0$.

Let P be a simplicial cell ball. We define the simplicial poset SP by

$$SP = P \cup C(\partial P) / \sim$$

where \sim is the equivalence relation defined by $\sigma \sim (\sigma, 1)$ for all $\sigma \in \partial P$. In other words, SP is a simplicial cell sphere such that $\Gamma(SP)$ is obtained from $\Gamma(P)$ by coning off its boundary.

We need the next result due to Masuda [Ma, Section 5] (see also [MR] for a simplified proof).

Lemma 2.5 (Masuda). *Let P be a simplicial cell $(d-1)$ -sphere, $K[P] = A/I_P$ the face ring of P and $\Theta = \theta_1, \dots, \theta_d \in A_1$ an l.s.o.p. of $K[P]$. If $x_{v_1}x_{v_2} \cdots x_{v_d}$ is zero in $A/(I_P + (\Theta))$ for any sequence of d distinct vertices v_1, v_2, \dots, v_d of P then the number of the facets of P is even.*

In the rest of this section, we fix the following notation: Let P be a simplicial cell $(d-1)$ -ball and

$$\mathbf{h} = h(P) = (h_0, h_1, \dots, h_d).$$

Let $Q = C(\partial P)$ and $SP = P \cup C(\partial P) / \sim$ be simplicial posets defined as above. Also, let $A = K[x_\sigma : \sigma \in SP \setminus \{\hat{0}\}]$,

$$J = I_{SP} + (x_\sigma : \sigma \in SP \setminus P) = I_{SP} + (x_\sigma : \sigma \in Q \setminus \partial P),$$

$$L = I_{SP} + (x_\sigma : \sigma \in SP \setminus Q) = I_{SP} + (x_\sigma : \sigma \in P \setminus \partial P),$$

and let $\Theta = \theta_1, \theta_2, \dots, \theta_d \in A_1$ be a sequence of general linear forms. Then we have the following properties.

- $A/J \cong K[P]$, $A/L \cong K[Q]$ and

$$A/(L + J) = A/(I_{SP} + (x_\sigma : \sigma \in SP \setminus \partial P)) \cong K[\partial P].$$

- $\bar{\Theta} = \theta_1, \dots, \theta_{d-1}$ is an l.s.o.p. of $A/(L + J)$ and $\Theta = \theta_1, \dots, \theta_d$ is a common l.s.o.p. of A/I_{SP} , A/J and A/L .

Recall that $\sum_{k=0}^d h_k$ is equal to the number of the facets of P . The next lemma shows that, to prove (5)–(7), it is enough to study the number of the facets of SP .

Lemma 2.6. *Suppose $\partial h_i = 0$ for some $1 \leq i \leq d-2$. If the number of the facets of SP is even then $\sum_{k=0}^d h_k$ is even.*

Proof. By the construction of SP , the number of the facets of P is equal to that of SP minus that of ∂P . Then the statement follows from Theorem 1.1(3). \square

Lemma 2.7. *Let v_1, v_2, \dots, v_d be distinct vertices of SP . Then*

- If $\partial h_i = 0$ for some $1 \leq i \leq d-2$ then $x_{v_1}x_{v_2} \cdots x_{v_i} \in L + (\Theta)$.
- If $h_j = 0$ for some $1 \leq j \leq d$ then $x_{v_d}x_{v_{d-1}} \cdots x_{v_{d-j+1}} \in J + (\Theta)$.

Proof. Recall $A/J \cong K[P]$ and $A/L \cong K[Q]$. Since Q is the cone of ∂P ,

$$\dim_K (A/(L + (\Theta)))_k = h_k(Q) = h_k(\partial P) = \partial h_k$$

for $k = 0, 1, \dots, d-1$. Thus, $\partial h_i = 0$ implies $x_{v_1}x_{v_2} \cdots x_{v_i} \in L + (\Theta)$. Similarly, since $\dim_K (A/(J + (\Theta)))_j = h_j$, $h_j = 0$ implies $x_{v_d}x_{v_{d-1}} \cdots x_{v_{d-j+1}} \in J + (\Theta)$. \square

Now we prove (5) and (6).

Theorem 2.8. *If $\partial h_i = 0$ and $h_j = 0$ for some positive integers i and j with $i+j \leq d$ then $\sum_{k=0}^d h_k$ is even.*

Proof. Let v_1, v_2, \dots, v_d be distinct vertices of SP . By Lemmas 2.5 and 2.6, it is enough to prove that the monomial $x_{v_1}x_{v_2} \cdots x_{v_d}$ is zero in $A/(I_{SP} + (\Theta))$.

Since $i + j \leq d$, Lemma 2.7 shows

$$x_{v_1}x_{v_2} \cdots x_{v_d} \in LJ + (\Theta) \subset I_{SP} + (\Theta) + (x_\sigma x_\tau : \sigma \in P \setminus \partial P, \tau \in Q \setminus \partial P).$$

However, for all $\sigma \in P \setminus \partial P$ and $\tau \in Q \setminus \partial P$, we have $x_\sigma x_\tau \in I_{SP}$ since σ and τ have no common upper bounds. Thus $x_{v_1}x_{v_2} \cdots x_{v_d} \in I_{SP} + (\Theta)$. \square

Theorem 2.9. *Suppose $\partial h_n = 0$ for some $1 \leq n < \frac{d}{2}$. If $(h_\ell + h_{\ell-1} + \cdots + h_{\ell-n+1}) - \partial h_\ell < n$ for some $n \leq \ell \leq d - n$ then $\sum_{k=0}^d h_k$ is even.*

Proof. Let v_1, v_2, \dots, v_d be distinct vertices of SP . Then $x_{v_1}x_{v_2} \cdots x_{v_n} \in L + (\Theta)$ by Lemma 2.7(i). Since $n + \ell \leq d$, by the same argument as in the proof of Theorem 2.8, it is enough to prove that

$$x_{v_d}x_{v_{d-1}} \cdots x_{v_{d-\ell+1}} \in J + (\Theta).$$

Consider the exact sequence

$$0 \longrightarrow N \longrightarrow A/(J + (\bar{\Theta}, \theta_d^n)) \longrightarrow A/(L + J + (\bar{\Theta}, \theta_d^n)) \longrightarrow 0,$$

where $N = (L + J + (\bar{\Theta}, \theta_d^n))/(J + (\bar{\Theta}, \theta_d^n))$. Since $A/(L + J) \cong K[\partial P]$ and since $\bar{\Theta}$ is an l.s.o.p. of $A/(L + J)$, we have $\dim_K (A/(L + J + (\bar{\Theta})))_n = \partial h_n = 0$ and

$$A_n = (L + J + (\bar{\Theta}))_n.$$

In particular, we have $L + J + (\bar{\Theta}) = L + J + (\bar{\Theta}, \theta_d^n)$. Then, by Lemma 2.3 we have

$$\begin{aligned} (2.1) \quad \dim_K N_\ell &= \dim_K (A/(J + (\bar{\Theta}, \theta_d^n)))_\ell - \dim_K (A/(L + J + (\bar{\Theta}, \theta_d^n)))_\ell \\ &= (h_\ell + h_{\ell-1} + \cdots + h_{\ell-n+1}) - \partial h_\ell \\ &\leq n - 1, \end{aligned}$$

where the last inequality follows from the assumption.

Let $u_k = x_{v_d}x_{v_{d-1}} \cdots x_{v_{d-k+1}}$ for $k = 1, 2, \dots, \ell$. Since $A_n = (L + J + (\bar{\Theta}))_n$, any element in A of degree n is contained in $L + J + (\bar{\Theta})$. Since $\ell \geq n$, this fact says that the elements

$$u_\ell, u_{\ell-1}\theta_d, \dots, u_{\ell-n+1}\theta_d^{n-1}$$

are contained in $L + J + (\overline{\Theta})$ since they are products of elements of degree 1. Then (2.1) says that the above n elements are K -linearly dependent in N . Thus there are $\alpha_0, \alpha_1, \dots, \alpha_{n-1} \in K$ with $(\alpha_0, \alpha_1, \dots, \alpha_{n-1}) \neq (0, 0, \dots, 0)$ such that

$$\alpha_0 u_\ell + \alpha_1 u_{\ell-1} \theta_d + \dots + \alpha_{n-1} u_{\ell-n+1} \theta_d^{n-1} \in J + (\overline{\Theta}, \theta_d^n).$$

Let $s = \min\{k : \alpha_k \neq 0\}$. Then there is an $h \in A$ such that

$$\theta_d^s \alpha_s u_{\ell-s} + \theta_d^{s+1} h \in J + (\overline{\Theta}).$$

Since Θ is an l.s.o.p. of a Cohen-Macaulay ring A/J , θ_d is a non-zero divisor of $A/(J + (\overline{\Theta}))$. Thus

$$\alpha_s u_{\ell-s} + \theta_d h \in J + (\overline{\Theta}).$$

This fact implies $u_{\ell-s} \in J + (\Theta)$. Since $u_{\ell-s}$ divides u_ℓ , we have

$$x_{v_d} x_{v_{d-1}} \cdots x_{v_{d-\ell+1}} = u_\ell \in J + (\Theta),$$

as desired. \square

Next, we study the condition (7). To simplify the notation, we write $R = A/J \cong K[P]$ and let

$$C = (L + J)/J = ((x_\sigma : \sigma \in P \setminus \partial P) + J)/J.$$

Thus C is the ideal of R generated by the interior faces of P .

Lemma 2.10. *C is a d -dimensional Cohen-Macaulay A -module such that $H_C(t) = (h_d + h_{d-1}t + \dots + h_0t^d)/(1-t)^d$.*

Proof. Consider the exact sequence

$$0 \longrightarrow C \longrightarrow R \longrightarrow R/C \longrightarrow 0.$$

Observe $R \cong K[P]$ and $R/C = A/(L + J) \cong K[\partial P]$. Then

$$H_C(t) = H_R(t) - H_{R/C}(t) = H_{K[P]}(t) - H_{K[\partial P]}(t) = \frac{(h_d + h_{d-1}t + \dots + h_0t^d)}{(1-t)^d}.$$

Also, C is a d -dimensional Cohen-Macaulay A -module since R/C is a $(d-1)$ -dimensional Cohen-Macaulay A -module and R is a d -dimensional Cohen-Macaulay A -module (e.g., use [Ei, Corollary 18.6]). \square

Remark 2.11. The ideal C is the canonical module of R (see [BH, Chapter 3]). This fact can be proved in the same way as in the proof of [BH, Theorem 5.7.1], and gives an alternative proof of Lemma 2.10.

Since C is a submodule of R with $\dim C = \dim R$ and since C is Cohen-Macaulay, Θ is an l.s.o.p. of C and we have

$$\dim_K (C/(\Theta C))_i = h_{d-i}$$

for $i = 0, 1, \dots, d$ by Lemma 2.10. This fact shows

Lemma 2.12. *If $h_j = 0$ for some $1 \leq j \leq d-1$ then $C_{d-j} = \Theta C_{d-j-1}$.*

Lemma 2.13. *Suppose $h_j = 0$ for some $1 \leq j \leq d-1$. If $\theta_d f \in (C + \overline{\Theta}R)_{d-j}$ then, for any linear form $l \in R_1$, one has $lf \in C + \overline{\Theta}R$.*

Proof. By Lemma 2.12, $\theta_d f \in \overline{\Theta}R + \theta_d C$. Thus there is $g \in C$ such that $\theta_d(f - g) \in \overline{\Theta}R$. Since θ_d is a non-zero divisor of $R/(\overline{\Theta}R)$, we have $f - g \in \overline{\Theta}R$ and $f \in C + \overline{\Theta}R$. Hence $lf \in C + \overline{\Theta}R$. \square

The next theorem completes the proof of the necessity of Theorem 1.2.

Theorem 2.14. *Suppose $\partial h_i = 0$ and $h_j = 0$ for some integers i and j with $0 < i < \frac{d}{2}$ and $d - i < j < d$. If $\partial h_\ell \leq \ell$ for some $\ell \leq d - j$ then $\sum_{k=0}^d h_k$ is even.*

Proof. Let v_1, v_2, \dots, v_d be distinct vertices of SP . Since $i < j$, in the same way as in the proof of Theorem 2.8, it is enough to prove that

$$x_{v_d} x_{v_{d-1}} \cdots x_{v_{j+1}} \in J + (\Theta).$$

Let $u_k = x_{v_d} x_{v_{d-1}} \cdots x_{v_{d-k+1}}$ for $k = 0, 1, \dots, d - j$, where $u_0 = 1$. We regard each u_k as an element in $R = A/J$. Recall that ℓ is an integer such that $\ell \leq d - j$ and $\partial h_\ell \leq \ell$. Since $\overline{\Theta}$ is an l.s.o.p. of $R/C = A/(L + J) \cong K[\partial P]$,

$$\dim_K (R/(C + \overline{\Theta}R))_\ell = \partial h_\ell \leq \ell.$$

Then the elements

$$u_\ell, u_{\ell-1}\theta_d, u_{\ell-2}\theta_d^2, \dots, u_0\theta_d^\ell \in R_\ell$$

are K -linearly dependent in $R/(C + \overline{\Theta}R)$. Thus there are $\alpha_0, \alpha_1, \dots, \alpha_\ell \in K$ with $(\alpha_0, \alpha_1, \dots, \alpha_\ell) \neq 0$ such that

$$\alpha_0 u_\ell + \alpha_1 u_{\ell-1} \theta_d + \cdots + \alpha_\ell u_0 \theta_d^\ell \in C + \overline{\Theta}R.$$

Let $s = \min\{k : \alpha_k \neq 0\}$ and $h = (\sum_{k=s+1}^{\ell} \alpha_k u_{\ell-k} \theta_d^k) / \theta_d^{s+1}$, where $h = 0$ if $s = \ell$. Then $\theta_d^s (\alpha_s u_{\ell-s} + \theta_d h) \in (C + \overline{\Theta}R)_\ell$ and

$$\theta_d^{d-j-(\ell-s)} (\alpha_s u_{\ell-s} + \theta_d h) \in (C + \overline{\Theta}R)_{d-j}.$$

By Lemma 2.13, we have $(u_{d-j}/u_{\ell-s})(\alpha_s u_{\ell-s} + \theta_d h) \in C + \overline{\Theta}R$. Hence

$$\alpha_s u_{d-j} \in (C + \overline{\Theta}R + \theta_d R)_{d-j} \subset \Theta R,$$

where the last inclusion follows from Lemma 2.12. Since $\alpha_s \neq 0$, u_{d-j} is contained in ΘR . This fact shows $u_{d-j} = x_{v_d} x_{v_{d-1}} \cdots x_{v_{j+1}} \in J + (\Theta)$ when we regard u_{d-j} as an element of A . \square

3. PROOF OF SUFFICIENCY

In this section, we prove the sufficiency of Theorem 1.2.

Since the proof is a bit long, we first give a short outline of the proof. The key tools of the proof are the two main lemmas outlined below.

- (I) In Lemma 3.6, we introduce a simple way to make a new simplicial cell ball from a given simplicial cell ball by gluing a pair of simplexes along their boundaries.
- (II) In Lemma 3.10, we construct a special simplicial cell ball whose boundary has a nice property.

Then we construct a simplicial cell ball with the desired h -vector as follows: First, starting from a simplex, we gradually make a larger simplicial cell ball by using (I) repeatedly; Then, at a certain point, we attach a simplicial cell ball in (II) to change the shape of the boundary; Finally, we make a simplicial cell ball with the desired h -vector by using (I) again. The first technique (I) appeared in [Ko]. Indeed, if $\partial \mathbf{h}$ is positive or $\sum_{k=0}^d h_k$ is odd, then the first technique is enough to prove the desired statement. However, we need (II) when $\partial \mathbf{h}$ has a zero entry and $\sum_{k=0}^d h_k$ is even.

Before proving the main lemmas, we first study what happens to the h -vector when we glue two simplicial posets. Two finite posets P and Q are said to be *isomorphic* if there is a bijection $f : P \rightarrow Q$, called an *isomorphism*, such that, for all $\sigma, \tau \in P$, $\sigma > \tau$ if and only if $f(\sigma) > f(\tau)$. We write $P \cong Q$ if P and Q are isomorphic.

Let P and Q be simplicial posets. Let $I \subset P$ and $J \subset Q$ be order ideals which are isomorphic as posets. For a given isomorphism $f : I \rightarrow J$, we define the equivalence relation \sim_f on $P \cup Q$ by $\sigma \sim_f f(\sigma)$ for all $\sigma \in I$, and define

$$P \cup_f Q = (P \cup Q) / \sim_f.$$

Then $P \cup_f Q$ is again a simplicial poset. We write $P \cup_I Q$ (or $P_I \cup_I Q$) for this simplicial poset $P \cup_f Q$ when the isomorphism f is clear.

Lemma 3.1. *Let P and Q be $(d-1)$ -dimensional simplicial posets and let $I \subset P$ and $J \subset Q$ be order ideals with $I \cong J$. If $\dim I = d-2$ then*

$$h_k(P \cup_I Q) = h_k(P) + h_k(Q) - h_k(I) + h_{k-1}(I)$$

for $k = 0, 1, \dots, d$, where $h_{-1}(I) = h_d(I) = 0$.

Proof. The desired formula follows from $f_k(P \cup_I Q) = f_k(P) + f_k(Q) - f_k(I)$ by straightforward computations. \square

For a positive integer i , we write $[i] = \{1, 2, \dots, i\}$. For integers $1 \leq k \leq d$, let

$$\Delta_d(k) = \{F \subset [d] : F \not\supset [k]\}.$$

By defining a partial order on $\Delta_d(k)$ by inclusion, we regard $\Delta_d(k)$ as a simplicial poset of dimension $d-2$. In other words, $\Delta_d(k)$ is the (abstract) simplicial complex generated by $\{[d] \setminus \{i\} : i = 1, 2, \dots, k\}$. Let $\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_d \in \mathbb{Z}^{d+1}$ be the unit vectors of \mathbb{Z}^{d+1} . Thus \mathbf{e}_i is the vector such that $(i+1)$ -th entry of \mathbf{e}_i is 1 and all other entries of \mathbf{e}_i are 0. It is easy to see that $h_i(\Delta_d(k)) = 1$ for $i < k$ and $h_i(\Delta_d(k)) = 0$ for $i \geq k$. Then the next lemma follows from Lemma 3.1.

Lemma 3.2. *With the same notation as in Lemma 3.1, if $I \cong \Delta_d(k)$ then*

$$h(P \cup_I Q) = h(P) + h(Q) - \mathbf{e}_0 + \mathbf{e}_k.$$

To apply Lemmas 3.1 and 3.2 efficiently, it is convenient to have a nice way to construct simplicial cell balls by gluing two simplicial cell balls. Constructibility and shellability give such nice ways.

Definition 3.3. A $(d-1)$ -dimensional simplicial poset P is said to be *constructible* if

- (i) P is a Boolean algebra of rank d , or
- (ii) there are $(d - 1)$ -dimensional constructible simplicial posets Q and Q' such that $P = Q_I \cup_{I'} Q'$ for some order ideals $I \subset Q$ and $I' \subset Q'$ and that $I \cong I'$ is a $(d - 2)$ -dimensional constructible simplicial poset.

Definition 3.4. A simplicial poset is said to be *pure* if all its facets have the same rank. A $(d - 1)$ -dimensional pure simplicial poset P is said to be *shellable* if there is an order $\sigma_1, \sigma_2, \dots, \sigma_r$ of the facets of P , called a *shelling of P* , such that, for $i = 2, 3, \dots, r$, $\langle \sigma_1, \sigma_2, \dots, \sigma_{i-1} \rangle \cap \langle \sigma_i \rangle$ is isomorphic to $\Delta_d(k_i)$ for some $1 \leq k_i \leq d$.

Note that shellable simplicial posets are constructible since $\Delta_d(k)$ is constructible. In the rest of this paper, a simplicial cell d -ball which is constructible (respectively shellable) will be called a *constructible d -ball* (respectively *shellable d -ball*).

We need the following well-known properties of constructibility and shellability. See e.g., [Bj2, Theorem 11.4 and Section 12].

- Let P be a simplicial cell sphere. If $Q \subsetneq P$ is a shellable simplicial poset with $\dim Q = \dim P$, then Q is a simplicial cell ball.
- Boolean algebras and $\Delta_d(k)$ with $k < d$ are shellable balls.
- Let P and Q be constructible d -balls, and let $I \subsetneq \partial P$ and $J \subsetneq \partial Q$ be order ideals with $I \cong J$. If I is a constructible $(d - 1)$ -ball then $P_I \cup_J Q$ is a constructible d -ball.

For a finite set X , we write $\#X$ for the cardinality of X . The next well-known lemma, which immediately follows from Lemma 3.2 and the fact that a Boolean algebra has the h -vector $(1, 0, \dots, 0)$, is useful to compute h -vectors of shellable simplicial posets.

Lemma 3.5. *Let P be a $(d - 1)$ -dimensional shellable simplicial poset with a shelling $\sigma_1, \sigma_2, \dots, \sigma_r$. Then, for $i = 1, 2, \dots, d$, one has*

$$h_i(P) = \#\{\ell \in \{2, 3, \dots, r\} : \langle \sigma_1, \sigma_2, \dots, \sigma_{\ell-1} \rangle \cap \langle \sigma_\ell \rangle \cong \Delta_d(i)\}.$$

Now we prove our first main lemma. We say that a simplicial poset P contains a simplicial poset Q if there is an order ideal $I \subset P$ which is isomorphic to Q .

Lemma 3.6. *Fix integers $1 \leq i \leq m \leq d$ and $1 \leq j \leq d - i$. Let P be a constructible $(d - 1)$ -ball such that ∂P contains $\Delta_d(m)$ and let $\ell = \min\{m, i + j\}$. Then there is a constructible $(d - 1)$ -ball Q such that*

- (i) $h(Q) = h(P) + \mathbf{e}_i + \mathbf{e}_j$, and
- (ii) ∂Q contains $\Delta_d(\ell)$.

Proof. By the assumption, there is an order ideal $L \subset \partial P$ which is isomorphic to $\Delta_d(m)$. We write

$$L = \{\mathcal{L}(F) : F \subset [d], F \not\supset [m]\},$$

where the ordering is defined by $\mathcal{L}(F) > \mathcal{L}(G)$ if $F \supset G$. Since $i \leq m$,

$$I = \{\mathcal{L}(F) : F \subset [d], F \not\supset [i]\} \cong \Delta_d(i)$$

is an order ideal of ∂P . We prepare two Boolean algebras

$$A = \{\mathcal{A}(F) : F \subset [d]\} \text{ and } B = \{\mathcal{B}(F) : F \subset [d]\},$$

where the ordering is defined by inclusion on F . Consider the order ideal

$$J = \{\mathcal{A}(F) : F \subset [d], F \not\ni \{i+1, \dots, i+j\}\} \cong \Delta_d(j).$$

Let

$$Q = (P \cup_I A) \cup_J B = (P \cup A \cup B) / \sim,$$

where \sim is the equivalence relation defined by $\mathcal{L}(F) \sim \mathcal{A}(F)$ if $F \not\ni [i]$ and $\mathcal{A}(F) \sim \mathcal{B}(F)$ if $F \not\ni \{i+1, \dots, i+j\}$. Then Q is a constructible $(d-1)$ -ball and has the desired h -vector by Lemma 3.2.

It remains to prove that ∂Q contains $\Delta_d(\ell)$. Clearly, ∂Q contains an order ideal

$$\Delta = \{\mathcal{B}(F) : F \subset [d], F \not\ni [i]\} \cup \{\mathcal{L}(F) : F \subset [d], F \not\ni \{i+1, \dots, \ell\}\} / \sim.$$

We claim that $\Delta \cong \Delta_d(\ell)$. What we must prove is that if $F \not\ni [i]$ and $F \not\ni \{i+1, \dots, \ell\}$ then $\mathcal{L}(F) \sim \mathcal{B}(F)$. Indeed, $F \not\ni [i]$ implies $\mathcal{L}(F) \sim \mathcal{A}(F)$. Also, since $\ell \leq i+j$, $F \not\ni \{i+1, \dots, \ell\}$ implies $\mathcal{A}(F) \sim \mathcal{B}(F)$. \square

We note that Lemma 3.6 essentially appeared in the proof of [Ko, Theorem 16].

Next, we develop a way to construct simplicial cell balls by applying Lemma 3.6 repeatedly. Let $\mathbf{h} = (h_0, h_1, \dots, h_d) \in \mathbb{Z}_{\geq 0}^{d+1}$ be a vector such that $\sum_{k=0}^d h_k$ is even. We write $\mathbf{h} = \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \dots + \mathbf{e}_{i_a}$, where $a = \sum_{k=0}^d h_k$ and where $i_1 \leq i_2 \leq \dots \leq i_a$. Consider the following decomposition of \mathbf{h} :

$$\mathbf{h} = (\mathbf{e}_{i_1} + \mathbf{e}_{i_a}) + (\mathbf{e}_{i_2} + \mathbf{e}_{i_{a-1}}) + \dots + (\mathbf{e}_{i_{\frac{a}{2}}} + \mathbf{e}_{i_{\frac{a}{2}+1}}).$$

We define the *initial number* $\text{in}(\mathbf{h})$ of \mathbf{h} and the *width* $\text{width}(\mathbf{h})$ of \mathbf{h} by

$$\text{in}(\mathbf{h}) = i_{\frac{a}{2}}$$

and

$$\text{width}(\mathbf{h}) = \min\{i_k + i_{a-k+1} : k = 1, 2, \dots, \frac{a}{2}\}.$$

The following fact is straightforward.

Lemma 3.7. *With the same notation as above, one has*

$$\text{width}(\mathbf{h}) = \max\{\ell : h_0 + h_1 + \dots + h_k \leq h_d + h_{d-1} + \dots + h_{\ell-k} \text{ for all } k\}.$$

We need the following reformulation of Lemma 3.6.

Lemma 3.8. *Let $\mathbf{h} = (h_0, h_1, \dots, h_d) \in \mathbb{Z}_{\geq 0}^{d+1}$ be such that $h_0 = 0$, $\partial h_i \geq 0$ for $i = 0, 1, \dots, d-1$ and $\sum_{k=0}^d h_k$ is even. Let P be a constructible $(d-1)$ -ball such that ∂P contains $\Delta_d(\ell)$ with $\ell \geq \text{in}(\mathbf{h})$. Then there is a constructible $(d-1)$ -ball Q such that $h(Q) = h(P) + \mathbf{h}$ and ∂Q contains $\Delta_d(\min\{\ell, \text{width}(\mathbf{h})\})$.*

Proof. We write

$$\mathbf{h} = (\mathbf{e}_{i_1} + \mathbf{e}_{i_a}) + (\mathbf{e}_{i_2} + \mathbf{e}_{i_{a-1}}) + \dots + (\mathbf{e}_{i_{\frac{a}{2}}} + \mathbf{e}_{i_{\frac{a}{2}+1}}),$$

where $a = \sum_{k=0}^d h_k$ and where $i_1 \leq i_2 \leq \dots \leq i_a$. For $k = 1, 2, \dots, \frac{a}{2}$, let

$$w_k = \min\{\ell, i_1 + i_a, i_2 + i_{a-1}, \dots, i_k + i_{a-k+1}\}$$

and let $w_0 = \ell$. The assumption $\partial h_i \geq 0$ for $i = 0, 1, \dots, d-1$ implies that

$$(3.1) \quad i_l + i_{a-l+1} \leq d$$

for all $l = 1, 2, \dots, \frac{a}{2}$.

We inductively prove that, for $k = 0, 1, \dots, \frac{a}{2}$, there is a constructible $(d-1)$ -ball Q_k which satisfies the following two conditions:

- (i) $h(Q_k) = h(P) + (\mathbf{e}_{i_1} + \mathbf{e}_{i_a}) + (\mathbf{e}_{i_2} + \mathbf{e}_{i_{a-1}}) + \dots + (\mathbf{e}_{i_k} + \mathbf{e}_{i_{a-k+1}})$;
- (ii) ∂Q_k contains $\Delta_d(w_k)$.

Since $w_{\frac{a}{2}} = \min\{\ell, \text{width}(\mathbf{h})\}$, if the above statement holds then $Q_{\frac{a}{2}}$ is the simplicial poset with the desired properties.

For $k = 0$, $Q_0 = P$ satisfies the desired conditions. Suppose that such Q_k , where $0 \leq k < \frac{a}{2}$, exists. We prove the existence of Q_{k+1} . Observe

$$i_{k+1} \leq \min\{\ell, i_{a-k}\} \leq w_k,$$

where the last inequality follows from $i_{a-k} \leq i_{a-k+1} \leq \dots \leq i_a$. Recall that $i_{a-k} \leq d - i_{k+1}$ by (3.1) and that ∂Q_k contains $\Delta_d(w_k)$ by the induction hypothesis. By Lemma 3.6 (apply the case when $P = Q_k$, $i = i_{k+1}$, $m = w_k$ and $j = i_{a-k}$) there is a constructible $(d-1)$ -ball Q_{k+1} such that $h(Q_{k+1}) = h(Q_k) + \mathbf{e}_{i_{k+1}} + \mathbf{e}_{i_{a-k}}$ and ∂Q_{k+1} contains $\Delta_d(\min\{w_k, i_{k+1} + i_{a-k}\}) = \Delta_d(w_{k+1})$ as desired. \square

By considering the special case when P is a Boolean algebra, we obtain the next corollary.

Corollary 3.9 (Kolins). *Let $\mathbf{h} = (h_0, h_1, \dots, h_d) \in \mathbb{Z}_{\geq 0}^{d+1}$ be such that $h_0 = 0$, $\partial h_i \geq 0$ for $i = 0, 1, \dots, d-1$ and $\sum_{k=0}^d h_k$ is even. Then there is a constructible $(d-1)$ -ball P such that $h(P) = \mathbf{e}_0 + \mathbf{h}$ and ∂P contains $\Delta_d(\text{width}(\mathbf{h}))$.*

The next lemma is our second main lemma.

Lemma 3.10. *Let n, m and d be positive integers such that $n \leq \frac{d}{2}$ and $d - n \leq m < d$. Let $\mathbf{h} = (h_0, h_1, \dots, h_d) \in \mathbb{Z}^{d+1}$ be a vector such that $\sum_{k=0}^d h_k = d$, $h_i = 1$ for $0 \leq i < d - n$, $h_i > 0$ for $d - n \leq i \leq m$ and $h_i = 0$ for $i > m$. There is a constructible $(d-1)$ -ball P such that*

- (i) $h(P) = \mathbf{h}$, and
- (ii) ∂P contains two ideals I_1 and I_2 that have no common facets such that $I_1 \cong \Delta_d(n)$ and $I_2 \cong \Delta_d(d-n)$.

Proof. Consider the simplicial posets

$$A = \{\mathcal{A}(F) : F \subset [d+1], F \not\supset [n]\} \cong \Delta_{d+1}(n)$$

and

$$B = \{\mathcal{B}(F) : F \subset [d+1], F \not\supset \{n+1, \dots, d\}\} \cong \Delta_{d+1}(d-n),$$

where the ordering is defined by inclusion on F . Consider the simplicial complex

$$\begin{aligned} \Sigma &= \langle [d+1] \setminus \{i, j\} : i \in [n], j \in \{n+1, \dots, d\} \rangle \\ &= \{F \subset [d+1] : F \subset [d+1] \setminus \{i, j\} \text{ for some } i \in [n] \text{ and } j \in \{n+1, \dots, d\}\}. \end{aligned}$$

Let

$$I_1 = \langle \mathcal{A}([d+1] \setminus \{i, d+1\}) : i = 1, 2, \dots, n \rangle \cong \Delta_d(n)$$

and

$$I_2 = \langle \mathcal{B}([d+1] \setminus \{i, d+1\}) : i = n+1, n+2, \dots, d \rangle \cong \Delta_d(d-n).$$

Then

$$\begin{aligned}\partial A &= \langle \mathcal{A}([d+1] \setminus \{i, j\}) : i = 1, 2, \dots, n, j = n+1, \dots, d+1 \rangle \\ &= \{ \mathcal{A}(F) : F \in \Sigma \} \cup I_1,\end{aligned}$$

and similarly

$$\partial B = \{ \mathcal{B}(F) : F \in \Sigma \} \cup I_2.$$

Let $\mathbf{g} = (g_0, g_1, \dots, g_d) = \mathbf{h} - \sum_{i=0}^{n-1} \mathbf{e}_i - \sum_{i=0}^{d-n-1} \mathbf{e}_i$. Observe that A and B are constructible $(d-1)$ -balls with $h(A) = \sum_{i=0}^{n-1} \mathbf{e}_i$ and $h(B) = \sum_{i=0}^{d-n-1} \mathbf{e}_i$. By Lemma 3.1, if there is a constructible $(d-2)$ -ball $\Omega \subset \Sigma$ such that $-h_i(\Omega) + h_{i-1}(\Omega) = g_i$ for all i , then the simplicial poset

$$P = A \cup_{\Omega} B = (A \cup B) / \sim,$$

where \sim is the equivalence relation defined by $\mathcal{A}(F) \sim \mathcal{B}(F)$ for $F \in \Omega$, satisfies the desired conditions (i) and (ii). Thus the next lemma completes the proof. \square

Lemma 3.11. *With the same notation as in the proof of Lemma 3.10, there is a shellable $(d-2)$ -ball $\Omega \subset \Sigma$ such that $-h_i(\Omega) + h_{i-1}(\Omega) = g_i$ for all i .*

Proof. Observe

$$\mathbf{g} = (-1, \dots, -1, 0, \dots, 0, h_{d-n}, h_{d-n+1}, \dots, h_d),$$

where -1 appears in the first n entries. Let

$$\alpha_{\ell} = h_{(d-n)+\ell-1} + h_{(d-n)+\ell} + \dots + h_d$$

for $\ell = 1, 2, \dots, n+1$. By the assumption on $\mathbf{h} = (h_0, h_1, \dots, h_d)$, $\alpha_1 = n$ and $\alpha_{\ell} \leq n - (\ell - 1)$ for all ℓ .

Let

$$\begin{aligned}D &= \{ \{p, q\} : p \in [n], q \in \{n+1, \dots, d\}, p+q \leq d \} \\ &\quad \bigcup \left[\bigcup_{\ell=1}^{m-(d-n)+1} \{ \{p, d+\ell-p\} : p = \ell, \ell+1, \dots, \ell + \alpha_{\ell} - 1 \} \right].\end{aligned}$$

Since $\alpha_{\ell} \leq n - (\ell - 1)$, $D \subset \{ \{p, q\} : p \in [n], q \in \{n+1, \dots, d\} \}$. Let

$$\Omega = \langle [d+1] \setminus \{p, q\} : \{p, q\} \in D \rangle \subset \Sigma.$$

We claim that Ω satisfies the desired conditions.

We first prove that Ω is shellable. We define the total order \succ on D by $\{p, q\} \succ \{p', q'\}$, where $p < q$ and $p' < q'$, if $p < p'$ or $p = p'$ and $q < q'$. We show that the total order on the facets of Ω induced by \succ gives a shelling of Ω . Since $h_{n-d}, \dots, h_m > 0$ and $h_{m+1} = \dots = h_d = 0$ by the assumption, we have $\alpha_1 > \alpha_2 > \dots > \alpha_{m-(d-n)+2} = \dots = \alpha_{n+1} = 0$. Then, by the construction of D , for any $\{p, q\} \in D$ with $p < q$, we have $\{p-1, q\} \in D$ if $p \neq 1$ and $\{p, q-1\} \in D$ if $q \neq n+1$. This fact shows that,

for any $\{p, q\} \in D$ with $\{p, q\} \neq \{1, n+1\}$,

$$\begin{aligned} & \langle [d+1] \setminus \{s, t\} : \{s, t\} \in D, \{s, t\} \succ \{p, q\} \rangle \cap \langle [d+1] \setminus \{p, q\} \rangle \\ &= \langle [d+1] \setminus \{p, q, k\} : k = 1, 2, \dots, p-1, n+1, n+2, \dots, q-1 \rangle \\ &\cong \Delta_{d-1}(p+q-n-2). \end{aligned}$$

Hence Ω is shellable. (To see that the second line contains the first line, use the fact that if F is contained in the first line then one has either $p' = \min([n] \setminus F) < p$ or $q' = \min(\{n+1, \dots, d\} \setminus F) < q$ since $[d+1] \setminus \{p', q'\}$ must be contained in $\{[d+1] \setminus \{s, t\} : \{s, t\} \in D, \{s, t\} \succ \{p, q\}\}$.)

The above shelling and Lemma 3.5 show

$$h_i(\Omega) = \#\{\{p, q\} \in D : p+q-n-2 = i\}$$

for $i = 0, 1, \dots, d-1$. Then a simple counting shows

$$h_i(\Omega) = \begin{cases} i+1, & \text{if } i \leq n-1, \\ n, & \text{if } n \leq i \leq d-n-1, \\ \alpha_{i-(d-n)+2}, & \text{if } i \geq d-n. \end{cases}$$

Note that $h_{d-n-1}(\Omega) = \alpha_1 = n$. Then we have $-h_i(\Omega) + h_{i-1}(\Omega) = g_i$ for all i .

Finally, Ω is a simplicial cell $(d-2)$ -ball since Ω can be identified with a $(d-2)$ -dimensional subsubset of a $(d-2)$ -dimensional simplicial cell sphere ∂A . \square

Remark 3.12. The simplicial complex Σ is the cone of the join of the boundaries of simplexes, and h -vectors of shellable subcomplexes of Σ having the same dimension as Σ are characterized in [BFS]. This will give an alternative proof of Lemma 3.11.

Theorem 3.13. *If $\mathbf{h} = (h_0, h_1, \dots, h_d) \in \mathbb{Z}^{d+1}$ satisfies conditions (1)–(7) in Theorem 1.2, then \mathbf{h} is the h -vector of a simplicial cell $(d-1)$ -ball.*

Proof. If $\partial \mathbf{h}$ is positive or $\sum_{k=0}^d h_k$ is even, then the assertion was proved in [Ko, Theorem 16]. But we include a proof for completeness (our proof is essentially the same as that of Kolins).

Case 1. Suppose that all entries of $\partial \mathbf{h}$ are positive. If $\sum_{k=0}^d h_k$ is odd then, since $\partial(\mathbf{h} - \mathbf{e}_0)$ is a non-negative vector, Corollary 3.9 says that $\mathbf{h} = \mathbf{e}_0 + (\mathbf{h} - \mathbf{e}_0)$ is the h -vector of a simplicial cell $(d-1)$ -ball. Suppose that $\sum_{k=0}^d h_k$ is even. We write

$$\mathbf{h} - \mathbf{e}_0 = (\mathbf{e}_{i_1} + \mathbf{e}_{i_a}) + (\mathbf{e}_{i_2} + \mathbf{e}_{i_{a-1}}) + \cdots + (\mathbf{e}_{i_{\frac{a-1}{2}}} + \mathbf{e}_{i_{\frac{a+3}{2}}}) + \mathbf{e}_{i_{\frac{a+1}{2}}},$$

where $a = \sum_{k=0}^d h_k - 1$ and where $i_1 \leq \cdots \leq i_a$. Let $\mathbf{h}' = \mathbf{h} - \mathbf{e}_0 - \mathbf{e}_{i_{\frac{a+1}{2}}}$. Since $\partial(\mathbf{h} - \mathbf{e}_0)$ is non-negative, we have $i_{a-l+1} \leq d - i_l$ for $l = 1, 2, \dots, \frac{a-1}{2}$. Then $\partial \mathbf{h}'$ is non-negative and $\text{width}(\mathbf{h}') \geq \min\{i_{\frac{a+3}{2}}, \dots, i_a\} \geq i_{\frac{a+1}{2}}$. Thus, by Corollary 3.9, there is a constructible $(d-1)$ -ball P such that $h(P) = \mathbf{e}_0 + \mathbf{h}' = \mathbf{h} - \mathbf{e}_{i_{\frac{a+1}{2}}}$ and ∂P contains $\Delta_d(i_{\frac{a+1}{2}})$. Let Q be a simplicial cell $(d-1)$ -ball obtained from P by gluing a Boolean algebra of rank d along an order ideal $J \subset \partial P$ with $J \cong \Delta_d(i_{\frac{a+1}{2}})$. Lemma 3.2 guarantees that the h -vector of Q is $h(P) + \mathbf{e}_{i_{\frac{a+1}{2}}} = \mathbf{h}$.

Case 2. Suppose that $\partial \mathbf{h}$ has a zero entry and $\sum_{k=0}^d h_k$ is even. Let

$$n = \min\{k : \partial h_k = 0\}$$

and let $\mathbf{h}' = (h'_0, h'_1, \dots, h'_d) = \mathbf{h} - \mathbf{e}_0 - \mathbf{e}_{d-n}$. Note that \mathbf{h}' is a non-negative vector since $\partial h_{n-1} > 0$ and $\partial h_n = 0$ imply $h_{d-n} > 0$. By the same argument as in Case 1, it is enough to prove that there is a constructible $(d-1)$ -ball P such that $h(P) = \mathbf{h} - \mathbf{e}_{d-n}$ and ∂P contains $\Delta_d(d-n)$. By the choice of n , $\partial \mathbf{h}'$ is non-negative. Thus, by Corollary 3.9, what we must prove is $\text{width}(\mathbf{h}') \geq d-n$.

By the condition (4), for $k \geq 0$, we have

$$h_{k+n} + \dots + h_{k+1} \geq (h_0 + h_1 + \dots + h_{k+n}) - (h_d + h_{d-1} + \dots + h_{d-k-n})$$

(we substitute k by $k+n$ to the inequality in (4)). Then we have $h_0 + \dots + h_k \leq h_d + \dots + h_{d-k-n}$ for all $k \geq 0$. This fact shows

$$h'_0 + h'_1 + \dots + h'_k \leq h'_d + h'_{d-1} + \dots + h'_{d-k-n}$$

for all $k \geq 0$. Then Lemma 3.7 guarantees $\text{width}(\mathbf{h}') \geq d-n$.

Case 3. Suppose that $\partial \mathbf{h}$ has a zero entry and $\sum_{k=0}^d h_k$ is odd. By the condition (3), d must be even. Let

$$n = \min\{k : \partial h_k = 0\}.$$

By the symmetry of $\partial \mathbf{h}$, we have $n < \frac{d}{2}$. Let

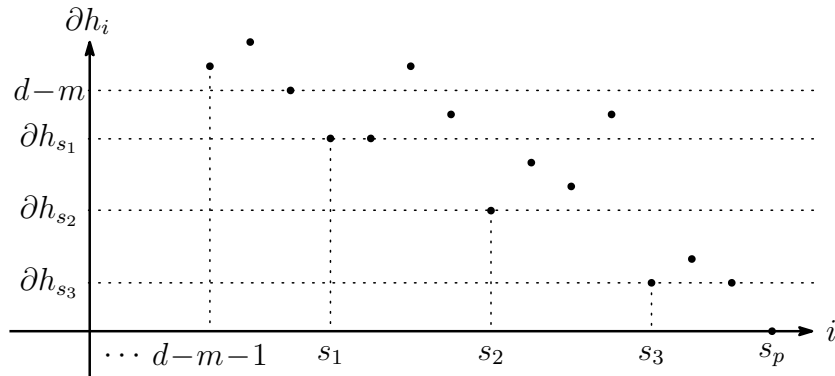
$$m = \min\{k : h_k = 0\} - 1.$$

Thus $h_0, h_1, \dots, h_m > 0$ and $h_{m+1} = 0$. Then $m \leq d-1$ since $h_d = 0$ and $m \geq d-n$ by the condition (5).

By the condition (7), $\partial h_{d-m-1} \geq d-m$. Recall that $\partial h_n = 0$, $d-m \leq n$ and $\partial h_k > 0$ for $k < n$. Then, there is a sequence of integers $d-m \leq s_1 < s_2 < \dots < s_p = n$ such that

- $d-m > \partial h_{s_1} > \dots > \partial h_{s_p} = 0$, and
- for any $s_{i-1} < j < s_i$ with $i \geq 2$ one has $\partial h_j \geq \partial h_{s_{i-1}}$, and for any $d-m-1 < j < s_1$ one has $\partial h_j \geq d-m$.

(The next figure explains what s_1, s_2, \dots, s_p are.)



Let

$$\gamma = (d - m - \partial h_{s_1})\mathbf{e}_{d-s_1} + \sum_{j=2}^p (\partial h_{s_{j-1}} - \partial h_{s_j})\mathbf{e}_{d-s_j}$$

and

$$\boldsymbol{\delta} = (\delta_0, \delta_1, \dots, \delta_d) = \sum_{i=0}^m \mathbf{e}_i + \gamma.$$

By the construction of $\boldsymbol{\delta}$, $\sum_{k=0}^d \delta_k = d + 1$ and $\delta_{d-n} \geq 2$. Let

$$\bar{\boldsymbol{\delta}} = \boldsymbol{\delta} - \mathbf{e}_{d-n}.$$

Then $\bar{\boldsymbol{\delta}}$ satisfies the assumption of Lemma 3.10. Thus there is a constructible $(d-1)$ -ball P such that

- $h(P) = \bar{\boldsymbol{\delta}}$.
- ∂P contains two ideals I_1 and I_2 that have no common facets such that $I_1 \cong \Delta_d(n)$ and $I_2 \cong \Delta_d(d-n)$.

We define $\mathbf{h}' = (h'_0, h'_1, \dots, h'_d) \in \mathbb{Z}^{d+1}$ and $\mathbf{h}'' = (h''_0, h''_1, \dots, h''_d) \in \mathbb{Z}^{d+1}$ by

$$\mathbf{h}' = (0, \dots, 0, h_{n+1} - 1, \dots, h_{d-n-1} - 1, 0, \dots, 0)$$

and

$$\mathbf{h}'' = (h_0 - 1, \dots, h_n - 1, 0, \dots, 0, h_{d-n} - 1, \dots, h_m - 1, h_{m+1}, \dots, h_d) - \gamma.$$

We claim that the following conditions hold.

- (i) $\mathbf{h} = \mathbf{h}' + \mathbf{h}'' + \bar{\boldsymbol{\delta}} + \mathbf{e}_{d-n}$;
- (ii) \mathbf{h}' and \mathbf{h}'' are non-negative;
- (iii) $\sum_{k=0}^d h'_k$ and $\sum_{k=0}^d h''_k$ are even;
- (iv) $\partial \mathbf{h}'$ and $\partial \mathbf{h}''$ are non-negative and $\partial h''_n = 0$;
- (v) $\text{width}(\mathbf{h}') \geq d - n$.

We first prove that the above conditions (i)–(v) prove the desired statement.

By Corollary 3.9 and conditions (ii)–(v), there is a constructible $(d-1)$ -ball Q with $h(Q) = \mathbf{e}_0 + \mathbf{h}'$ such that ∂Q contains an order ideal J which is isomorphic to $\Delta_d(d-n)$. Then the simplicial poset

$$R = Q_J \cup_{I_2} P$$

is a constructible $(d-1)$ -ball with $h(R) = \mathbf{h}' + \bar{\boldsymbol{\delta}} + \mathbf{e}_{d-n}$ by Lemma 3.2. Also, ∂R contains $\Delta_d(n)$ since I_1 and I_2 have no common facets.

Since \mathbf{h}'' is non-negative and $\sum_{k=0}^d h''_k$ is even, \mathbf{h}'' can be written in the form

$$\mathbf{h}'' = (\mathbf{e}_{i_1} + \mathbf{e}_{i_a}) + (\mathbf{e}_{i_2} + \mathbf{e}_{i_{a-1}}) + \dots + (\mathbf{e}_{i_{\frac{a}{2}}} + \mathbf{e}_{i_{\frac{a}{2}+1}})$$

where $a = \sum_{k=0}^d h''_k$ and $i_1 \leq \dots \leq i_a$. Since $\partial h''_n = 0$, we have

$$i_1 \leq \dots \leq i_{\frac{a}{2}} \leq n \text{ and } d - n \leq i_{\frac{a}{2}+1} \leq \dots \leq i_a.$$

Thus $\text{in}(\mathbf{h}'') \leq n$. Since ∂R contains $\Delta_d(n)$, Lemma 3.8 guarantees the existence of a constructible $(d-1)$ -ball whose h -vector is $h(R) + \mathbf{h}'' = \mathbf{h}$.

It remains to prove (i)–(v). Statement (i) is obvious.

Proof of (ii). Since $h_k > 0$ for $k \leq d - n$ (see the first paragraph of Case 3), \mathbf{h}' is non-negative. To prove that \mathbf{h}'' is non-negative, it is enough to prove that $\delta_k \leq h_k$ for all k . For $k > m$, $h_k \geq 0 = \delta_k$. Also, if $k \leq m$ and $k \notin \{d - s_1, \dots, d - s_p\}$ then $h_k \geq 1 = \delta_k$ by the choice of m . Suppose $k = d - s_j$. Since $h_{s_j} - h_{d-s_j} = \partial h_{s_j} - \partial h_{s_j-1}$,

$$h_{d-s_j} = h_{s_j} + (\partial h_{s_j-1} - \partial h_{s_j}) \geq 1 + (\partial h_{s_j-1} - \partial h_{s_j}) = \delta_{d-s_j},$$

where the inequality follows from the choice of s_1, \dots, s_p . (If $j = 1$ then we consider that $\partial h_{s_0} = d - m$.)

Proof of (iii). Since $\partial h_n = (h_0 + \dots + h_n) - (h_d + \dots + h_{d-n}) = 0$ and since $h_0 + h_1 + \dots + h_d$ is odd,

$$h_{n+1} + \dots + h_{d-n-1} = (h_0 + \dots + h_d) - (h_0 + \dots + h_n) - (h_d + \dots + h_{d-n})$$

is odd. Then, since d is even,

$$(h'_0 + \dots + h'_d) = (h_{n+1} + \dots + h_{d-n-1}) - (d - 2n - 1)$$

is even, as desired. Similarly, $\partial h_n = 0$ implies that

$$(h''_0 + \dots + h''_d) = (h_0 + \dots + h_n) + (h_d + \dots + h_{d-n}) - 2n - 2$$

is even.

Proof of (iv). Observe $\partial h'_k = \partial h_k - \partial h_n$ for $k \geq n$. Since $\partial h_n = 0$, we have

$$\partial h'_k = \begin{cases} 0, & \text{for } k = 0, 1, \dots, n, \\ \partial h_k, & \text{for } k = n + 1, \dots, \frac{d}{2} - 1. \end{cases}$$

Hence $\partial \mathbf{h}'$ is non-negative.

For $k = 0, 1, \dots, d - m - 1$,

$$\partial h''_k = \partial h_k - (k + 1) \geq 0$$

by the condition (7). Also, for $k = n, \dots, \frac{d}{2} - 1$,

$$\partial h''_k = \partial h_n = 0.$$

Let $k \in \{d - m, d - m + 1, \dots, n - 1\}$. Then, there is an i such that $s_{i-1} \leq k < s_i$, where we consider that $s_0 = d - m$ and $\partial h_{s_0} = d - m$. Then

$$\begin{aligned} \partial h''_k &= \partial h_k - (d - m) + \{(d - m - \partial h_{s_1}) + (\partial h_{s_1} - \partial h_{s_2}) + \dots + (\partial h_{s_{i-2}} - \partial h_{s_{i-1}})\} \\ &= \partial h_k - \partial h_{s_{i-1}} \\ &\geq 0, \end{aligned}$$

where the last inequality follows from the choice of s_1, \dots, s_p . Thus $\partial \mathbf{h}''$ is non-negative.

Proof of (v). The condition (6) says that, for $0 \leq k \leq d - 2n$,

$$\begin{aligned} h_{k+n} + \dots + h_{k+1} - n &\geq \partial h_{k+n} \\ &= \partial h_n + (h_{n+1} + \dots + h_{n+k}) - (h_{d-n-1} + \dots + h_{d-n-k}) \end{aligned}$$

(we substitute ℓ by $k + n$ in the inequality in (6)). Since $\partial h_n = 0$, for $n < k \leq d - 2n$, we have

$$h_{n+1} + \dots + h_k + n \leq h_{d-n-1} + \dots + h_{d-n-k}.$$

The above inequality says

$$\begin{aligned} h'_0 + \cdots + h'_k &= h_{n+1} + \cdots + h_k - (k - n) \\ &\leq h_{d-n-1} + \cdots + h_{d-n-k} - k = h'_d + \cdots + h'_{d-n-k} \end{aligned}$$

for all $n < k < d - 2n$. Also, if $k \leq n$ or $k \geq d - 2n$ then it is clear that

$$h'_0 + \cdots + h'_k \leq h'_d + \cdots + h'_{d-n-k}.$$

Then Lemma 3.7 guarantees $\text{width}(\mathbf{h}') \geq d - n$. □

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