.

MINIMAL BALANCED NEIGHBORLY POLYNOMIALS

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Dedicated with gratitude to our friend Ngo Viet Trung on the occasion of his 70th birthday

ABSTRACT. In this paper we introduce minimal balanced neighbourly polynomials and show some methods to construct such polynomials. In particular, using this notion, we prove the existence of balanced neighbourly polynomials of the following types: (i) type $(p,\dots,p)$ for most prime numbers p, (ii) types $(d-1,d,d,d)$, $(d-1,d-1,d,d)$ and $(d-1,d-1,d-1,d)$ when d is odd or is divisible by 4. We also construct balanced neighbourly simplicial spheres of type $(2,4k-1,4k-1,4k-1)$.

1. INTRODUCTION

Balanced neighbourly polynomials (BNPs for short) were introduced by the second author [13] motivated from an existence problem of balanced neighbourly simplicial spheres in the work of Zheng [14]. The main question on BNPs is the following: For which vector $(d_1,\dots,d_n)\in\mathbb{N}^n$, a BNP of type $(d_1,\dots,d_n)$ exists? At this moment, this problem has been mainly studied when $n=4$ and $d_1=\cdots=d_4$. In this paper, we introduce special classes of BNPs, which we call minimal BNPs and weakly minimal BNPs, and develop a way to determine their existence.

Let $\boldsymbol{d}=(d_1,\dots,d_n)\in\mathbb{N}^n$ and $S=S_{[\boldsymbol{d}]}=\Bbbk[x_{i,j}\mid 1\le i\le n,\ 1\le j\le d_i]$ the polynomial ring over a field $\Bbbk$ with the $\mathbb{Z}^n$-grading defined by $\deg x_{i,j}=\mathbf{e}_i$, where $\mathbf{e}_1,\dots,\mathbf{e}_n$ are standard vectors of $\mathbb{Z}^n$. Let $\partial_{i,j}=\frac{\partial}{\partial x_{i,j}}$ and for each polynomial $f\in S$, let

$$\operatorname{ann}^*(f)=\{g(x_{i,j})\in S\mid g(\partial_{i,j})f=0\}.$$

It is known that if $\operatorname{char}(\Bbbk)=0$ or f is a linear combination of squarefree monomials, then $S/\operatorname{ann}^*(f)$ is an Artinian Gorenstein algebra (see section 2). For a $\mathbb{Z}^n$-graded S-module M, let $M_{\boldsymbol{a}}$ denote the graded component of M of degree $\boldsymbol{a}\in\mathbb{Z}^n$ and let

$$H(M,\boldsymbol{a})=\dim_{\Bbbk}M_{\boldsymbol{a}}.$$

The function $H(M,-):\mathbb{Z}^n\to\mathbb{Z}$ is called the $\mathbb{Z}^n$-graded Hilbert function of M. To simplify notation, for $A\subset[n]=\{1,2,\dots,n\}$, we write $\mathbf{e}_A=\sum_{i\in A}\mathbf{e}_i$ and write $H(M,A)=H(M,\mathbf{e}_A)$. We say that a polynomial $f\in S$ is a **balanced neighbourly polynomial** of type $(d_1,\dots,d_n)$ if it satisfies the following conditions

We thank Ngo Viet Trung for making a chance that the authors collaborate. The main part of this work was done when the second author visited the Waseda university in 2022. We thank the university for their kind hospitality. The first author is partially supported by KAKENHI 21K0319. This research was funded by the fundamental research program of Hung Vuong University under the project grant No.19/2023/HĐKH.HV23-19.

(i) f is homogeneous of degree $(1, 1, \ldots, 1) \in \mathbb{Z}^n$, in other words, f is a linear combination of squarefree monomials of the form $x_{1,j_1} x_{2,j_2} \cdots x_{n,j_n}$.

(ii) For any subset $A \subset [n]$,

$$H(S/\mathrm{ann}^*(f), A) = \min\{\prod_{i \in A} d_i, \prod_{i \in [n] \setminus A} d_i\}.$$

(iii) $H(S/\mathrm{ann}^*(f), \{i\}) = d_i$ for $i = 1, 2, \ldots, n$.

We note that $\prod_{i \in A} d_i = 1$ when $A = \varnothing$.

The definition of a BNP is motivated from the work of Zheng [14] who introduce the notion of balanced neighbourly simplicial spheres (BNSSs for short). The existence of BNSSs is an interesting problem in a combinatorial study of face numbers of simplicial complexes since if a BNSS of type $(d, \ldots, d) \in \mathbb{Z}^n$ exists, then it must have the largest face numbers among all balanced simplicial $(n-1)$-spheres with $(d+1) \times n$ vertices (see section 3). A BNP, introduced in [13], may be regarded as an algebraic abstraction of a BNSS. Indeed, if a BNP f of type $(d_1, \ldots, d_n)$ exists, then the ring $S/\mathrm{ann}^*(f)$ has the largest Hilbert function among all Artinian Gorenstein algebras of the form S/I having socle degree $(1, 1, \ldots, 1)$. Also, it is known that if a BNSS of type $(d_1, \ldots, d_n)$ exists then a BNP of the same type exists over any field (see section 3).

At this moment, not much are known about the existence of BNPs and BNSSs, and they have been studied mainly when $(d_1, \ldots, d_n) = (d, d, d, d) \in \mathbb{N}^4$ [13, 14]. In this paper, we introduce minimal and weakly minimal BNPs, and give some criterion to prove their existence and non-existence. Using ideas of minimal BNPs, we prove the following results.

- We prove the existence of a minimal BNP of type $(p, p, \ldots, p) \in \mathbb{N}^n$ for most prime numbers p.
- We prove the existence of BNPs of type $(d-1, d, d, d)$, $(d-1, d-1, d, d)$ and $(d-1, d-1, d-1, d)$ when d is odd or is divisible by 4.
- We discuss a connection between our ideals and cross-polytopal CW complexes in [5]. Using this idea, we construct a BNSS of type $(2, 4k-1, 4k-1, 4k-1)$ for any $k > 0$.

This paper is organized as follows. In section 2, we explain fundamental properties of BNPs. In section 3, we explain how BNPs are related to BNSSs. In particular, we explain a concrete way to make a BNP from a BNSS. In section 4, we introduce some minimality conditions for BNPs, and study their existence. Using ideas developed in section 4, we study existence of BNPs of types $(d-1, d, d, d)$, $(d-1, d-1, d, d)$ and $(d-1, d-1, d-1, d)$ in section 5 and construct a BNSS of type $(2, 4k-1, 4k-1, 4k-1)$ in section 6. In section 7, we give a few remarks on special types of BNPs and BNSSs.

2. Foundations on balanced neighbourly polynomials

In this section, we explain basic properties of BNPs. We first recall Macaulay's inverse system.

2.1. **Macaulay's inverse system.** Fix positive integers $n \leq m$ and a field $\Bbbk$. Let $R = \Bbbk[x_1,\dots,x_m]$ be a $\mathbb{Z}^n$-graded polynomial ring with $\deg(x_i) \in \{\mathbf{e}_1,\dots,\mathbf{e}_n\}$ for $i=1,2,\dots,m$, and $R^* = \Bbbk[y_1,\dots,y_m]$ with $\deg y_i = \deg x_i$ for all i (later we actually consider the case $R = \Bbbk[x_{i,j}]$ and $R^* = \Bbbk[y_{i,j}]$). We define the action $\circ$ of R^* to R by linearly extending the operation

$$y_1^{a_1}\cdots y_m^{a_m} \circ x_1^{b_1}\cdots x_m^{b_m} = \begin{cases} x_1^{b_1-a_1}\cdots x_m^{b_m-a_m} & \text{if } b_k \geq a_k \text{ for all } k,\\ 0 & \text{otherwise.}\end{cases} \tag{2.1}$$

For a homogeneous polynomial $f = f(x_1,\dots,x_m) \in R$, let $\mathrm{ann}(f)$ be the ideal of R^* consisting of all elements that annihilates f, that is,

$$\mathrm{ann}(f) = \{g \in R^* \mid g \circ f = 0\}.$$

An Artinian $\mathbb{Z}^n$-graded algebra R^*/I, where I is an ideal of R^*, is said to be **Gorenstein** of socle degree $\boldsymbol{b} \in \mathbb{Z}^n$ if

$$\mathrm{socle}(R^*/I) = (R^*/I)_{\boldsymbol{b}} \cong \Bbbk,$$

where $\mathrm{socle}(R^*/I) = \{f \in R^*/I \mid y_i f = 0 \text{ for all } i\}$. The following result, which suggests that the correspondence $f \in R \to R^*/\mathrm{ann}(f)$ gives a one to one correspondence between polynomials in R and Artinian Gorenstein algebras of the form R^*/I (up to a scalar multiple of f), is known as the Macaulay correspondence (see [3, §21.1]).

Lemma 2.1. *With the same notation as above,*

(i) *for any homogeneous polynomial $f \in R$, the algebra $R^*/\mathrm{ann}(f)$ is an Artinian Gorenstein algebra of socle degree $\deg f$.*
(ii) *for any Artinian Gorenstein algebra R^*/I of socle degree $\boldsymbol{b}$, there is a polynomial $f \in R$ of degree $\boldsymbol{b}$ such that $I = \mathrm{ann}(f)$.*

We next explain how Hilbert functions of $R^*/\mathrm{ann}(f)$ can be computed from f. For a homogeneous polynomial $f \in R$ of degree $\boldsymbol{b}$ and $\boldsymbol{a} \in \mathbb{Z}^n_{\geq 0}$ with $\boldsymbol{a} \leq \boldsymbol{b}$, let

$$\mathrm{shadow}(f,\boldsymbol{a}) = \{\alpha \circ f \mid \ \alpha \in R^* \text{ is a monomial of degree } \boldsymbol{b}-\boldsymbol{a}\}$$

and

$$\begin{aligned} H(f,\boldsymbol{a}) &= \dim_{\Bbbk}\{g \circ f \mid g \in R^* \text{ is a polynomial of degree } \boldsymbol{b}-\boldsymbol{a}\}\\ &= \dim_{\Bbbk} \mathrm{span}_{\Bbbk}(\mathrm{shadow}(f,\boldsymbol{a})).\end{aligned}$$

The following fact is well-known.

Lemma 2.2. *Let $f \in R$ be a homogeneous polynomial of degree $\boldsymbol{b} \in \mathbb{Z}^n_{\geq 0}$. For $\boldsymbol{a} \in \mathbb{Z}^n_{\geq 0}$ with $\boldsymbol{a} \leq \boldsymbol{b}$, one has*

$$H(R^*/\mathrm{ann}(f),\boldsymbol{a}) = H(f,\boldsymbol{a}).$$

Proof. The action $\circ$ in (2.1) gives a non-degenerated bilinear form

$$R_{\boldsymbol{b}} \times R^*_{\boldsymbol{b}} \to \Bbbk, \quad (h,g) \mapsto g \circ h.$$

This induces a non-degenerated bilinear form

$$\{g \circ f \mid g \in R^* \text{ is a polynomial of degree } \boldsymbol{b}-\boldsymbol{a}\} \times (R^*/\mathrm{ann}(f))_{\boldsymbol{a}} \to \Bbbk$$

so we have an isomorphism of $\Bbbk$-vector spaces

$$\{g \circ f \mid g \in R^* \text{ is a polynomial of degree } \boldsymbol{b}-\boldsymbol{a}\} \cong \mathrm{Hom}_{\Bbbk}((R^*/\mathrm{ann}(f))_{\boldsymbol{a}}, \Bbbk) \cong (R^*/\mathrm{ann}(f))_{\boldsymbol{a}},$$

which proves the desired property. □

It is sometimes convenient to consider inverse systems without introducing R^*. We call a polynomial f **squarefree** if it is a linear combination of squarefree monomials. For a squarefree polynomial $f \in R$, one has $y_i \circ f = \frac{\partial}{\partial x_i} f$, so the ideal $\mathrm{ann}(f) \subset R^*$ is the same ideal as

$$\mathrm{ann}^*(f) = \{g(x_1,\dots,x_m) \in R \mid g(\tfrac{\partial}{\partial x_1},\dots,\tfrac{\partial}{\partial x_m}) f = 0\}$$

given in the introduction[1]. Also, if f is a squarefree polynomial of degree $\boldsymbol{b}$, then

$$\mathrm{shadow}(f,\mathbf{a}) = \{\partial_\alpha f \mid \alpha \in R \text{ is a monomial of degree } \boldsymbol{b}-\boldsymbol{a}\}, \tag{2.2}$$

where $\partial_\alpha = \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_k}}$ for $\alpha = x_{i_1} \cdots x_{i_k}$. We use this formulation later in sections 4 and 5.

2.2. **BNPs and maximal Hilbert functions.** Let $\boldsymbol{d} = (d_1,\dots,d_n) \in \mathbb{N}^n$, where $\mathbb{N}$ is the set of positive integers, and $S = S_{[\boldsymbol{d}]} = \Bbbk[x_{i,j} \mid 1 \le i \le n,\ 1 \le j \le d_i]$. For a subset A, we write $H(f,A) = H(f,\mathbf{e}_A)$ and $H(S/I,A) = H(S/I,\mathbf{e}_A)$. Recall that S is $\mathbb{Z}^n$-graded by $\deg(x_{i,j}) = \mathbf{e}_i \in \mathbb{Z}^n$ and $H(S,A) = \prod_{i \in A} d_i$. As we defined in the introduction, a polynomial $f \in S$ is called a BNP of type $\boldsymbol{d}$ if it satisfies the following conditions:

(i) f is homogeneous of degree $(1,1,\dots,1) \in \mathbb{Z}^n$.

(ii) For any subset $A \subset [n]$, one has

$$H(f,A) = \min\{\prod_{i \in A} d_i, \prod_{i \in [n]\setminus A} d_i\}.$$

(iii) $H(f,\{i\}) = d_i$ for $i = 1,2,\dots,n$.

Note that we changed conditions (ii) and (iii) using Lemma 2.2 comparing the definition given in the introduction. Also, when we consider BNPs it is harmless to assume $d_1 \le \cdots \le d_n$ and $d_n \le d_1 \times \cdots \times d_{n-1}$. We assume these conditions throughout the paper.

Remark 2.3. Our definition of BNPs (also that of BNSSs in section 3) is slightly different to that in [13, 14], where the condition (ii) is replaced with $H(f,A) = \prod_{i \in A} d_i$ for all $A \subset [n]$ with $|A| \le \frac{d}{2}$. We do this modification because the condition (ii) looks more natural to consider the case when $d_1,\dots,d_n$ are different, but when $d_1 = \cdots = d_n$ our definition coincides with the definitions given in [13, 14].

If $R = S/I$ is an Artinian Gorenstein algebra of socle degree $(1,1,\dots,1)$ then R has a vector space decomposition

$$R = \bigoplus_{A \subset [n]} R_{\mathbf{e}_A}$$

[1]If f is not squarefree, then $\mathrm{ann}^*(f)$ is not the same ideal as $\mathrm{ann}(f)$ but the conclusion of Lemma 2.1 still holds when $\mathrm{char}(\Bbbk) = 0$. See e.g. [8].

and we have the duality of Hilbert functions

$$H(R,A)=H(R,[n]\setminus A). \tag{2.3}$$

The condition (ii) in the definition of BNPs has the following meaning.

Lemma 2.4. *If $R=S/I$ is an Artinian Gorenstein graded algebra of socle degree $(1,1,\dots,1)$, then for any $A\subset[n]$ one has*

$$H(R,A)\le \min\{\prod_{i\in A} d_i,\prod_{i\in[n]\setminus A} d_i\}.$$

Proof. Observe $H(S,A)=\prod_{i\in A} d_i$. The statement follows from the duality of Hilbert functions (2.3) and the obvious inequality $H(R,A)\le H(S,A)$. □

So a BNP is a polynomial f such that the algebra $R=S/\operatorname{ann}^*(f)$ satisfies the equality in Lemma 2.4.

We list a few obvious criterion of BNPs. The following fact immediately follows from the duality of Hilbert functions (2.3).

Lemma 2.5. *Let f be a polynomial satisfying conditions (i) and (iii) in the definition of BNPs. Then f is a BNP of type $\boldsymbol{d}$ if and only if $H(f,A)=\prod_{i\in A} d_i$ for any $A\subset[n]$ with $\prod_{i\in A} d_i\le\prod_{i\in[n]\setminus A} d_i$.*

For a graded algebra $R=S/I$, since $H(S,A)=\prod_{i\in A} d_i$, the following conditions are equivalent.

- $H(R,A)=\prod_{i\in A} d_i$.
- $H(R,A)=H(S,A)$.
- I contains no elements of degree $\mathbf{e}_A$.

Also, when $I=\operatorname{ann}(f)$ for some polynomial f, then the above conditions are also equivalent to

- $\operatorname{span}_{\Bbbk}(\operatorname{shadow}(f,A))=S_{\mathbf{e}_A}$,

where $\operatorname{shadow}(f,A)=\operatorname{shadow}(f,\mathbf{e}_A)$. In particular, by the third condition, if $H(R,A)=\prod_{i\in A} d_i$ for some $A\subset[n]$ then we have $H(R,B)=\prod_{i\in B} d_i$ for all $B\subset A$. This implies the following refinement of the previous lemma.

Lemma 2.6. *Let f be as in Lemma 2.5. Then f is a BNP of type $\boldsymbol{d}$ if and only if $H(f,A)=\prod_{i\in A} d_i$ for all maximal subsets $A\subset[n]$ satisfying $\prod_{i\in A} d_i\le\prod_{i\in[n]\setminus A} d_i$.*

Below we give a few examples of BNPs with $n\le 3$. We note that when $n\le 3$ the condition (iii) automatically implies the condition (ii). When we consider examples, we write $x_{1,i}=x_i, x_{2,i}=y_i$, $x_{3,i}=z_i$ and so on to simplify notation.

Example 2.7. If $n = 2$ then $(d_1, d_2) = (d, d)$ since $d_1 \le d_2$ and $d_n \le d_1 \times \cdots \times d_{n-1}$. In this case

$$f = x_1y_1 + \cdots + x_dy_d$$

is a BNP of type (d, d). Indeed one can see that $\mathrm{shadow}(f, \{1\}) = \{x_1, \ldots, x_d\}$ and $\mathrm{shadow}(f, \{2\}) = \{y_1, \ldots, y_d\}$.

Example 2.8. Consider the case $n = 3$. Then $d_3 \le d_1 \times d_2$. In this case, take d_3 distinct monomials $m_1, \ldots, m_d \in \{x_iy_j \mid 1 \le i \le d_1,\ 1 \le j \le d_i\}$ such that $m_1, \ldots, m_{d_1}$ contain $x_1, \ldots, x_{d_1}$ and $m_1, \ldots, m_{d_2}$ contain $y_1, \ldots, y_{d_2}$, then the polynomial

$$f = \sum_{k=1}^{d_3} m_k z_k$$

is a BNP of type (d_1, d_2, d_3) since $\mathrm{shadow}(f, \{1\}) = \{x_1, \ldots, x_{d_1}\}, \ldots, \mathrm{shadow}(f, \{3\}) = \{z_1, \ldots, z_{d_3}\}$.

3. Connections to Combinatorics

In this section, we explain a connection between BNPs and combinatorial study of balanced simplicial complexes.

3.1. **Balanced neighbourly simplicial complexes.** A simplicial complex Δ on $V = \{y_1, \ldots, y_m\}$ is a collection of subsets of V satisfying

(i) $\sigma \in \Delta$ and $\tau \subset \sigma$ imply $\tau \in \Delta$, and
(ii) $\{y_i\} \in \Delta$ for $i = 1, 2, \ldots, m$.

An element of Δ is called a **face** of Δ and a maximal face (under inclusion) is called a **facet** of Δ. The **dimension** of a face $\sigma \in \Delta$ is $\dim \sigma = |\sigma| - 1$ and the dimension of Δ is $\dim \Delta = \max_{\sigma \in \Delta} \dim \sigma$. We write $\Delta_k = \{\sigma \in \Delta \mid \dim \sigma = k\}$ the set of k-dimensional faces of Δ. A **proper** k**-coloring** of Δ is a map $c : V \to [k]$ satisfying $c(y_i) \ne c(y_j)$ for every edge $\{y_i, y_j\} \in \Delta$. A simplicial complex Δ of dimension $n-1$ is said to be **balanced** (completely balanced in some literature) if it admits a proper n-coloring. Also, a simplicial complex Δ is said to be a **simplicial** $(n-1)$**-sphere** if its geometric realization is homeomorphic to a $(n-1)$-sphere.

Fundamental combinatorial invariants of simplicial complexes are f- and h-vectors. Let Δ be an $(n-1)$-dimensional simplicial complex on V. Let $f_{k-1}(\Delta) = |\{\sigma \in \Delta \mid |\sigma| = k\}|$ for all k, where $f_{-1}(\Delta) = 1$. The vector $f(\Delta) = (f_{-1}(\Delta), f_0(\Delta), \ldots, f_{n-1}(\Delta))$ is called the f**-vector** of Δ. Also, the h**-vector** $h(\Delta) = (h_0(\Delta), h_1(\Delta), \ldots, h_n(\Delta))$ of Δ is defined by

$$h_i(\Delta) = \sum_{k=0}^{i} (-1)^{i-k} \binom{d-k}{i-k} f_{k-1}(\Delta) \quad \text{for } i = 0, 1, \ldots, n.$$

Now assume that Δ is balanced with a proper n-coloring $c : V \to [n]$. Then Δ has refinements of f- and h-vectors, called **flag** f**-vector** $(f_A(\Delta) \mid A \subset [n])$ and **flag** h**-vector**[2] $(h_A(\Delta) \mid$

[2]Flag f-vectors and flag h-vectors may depend on the choice of a proper coloring c, but when we discuss a balanced simplicial complexes we fix its coloring c.

$A \subset [n]$) defined by

$$f_A(\Delta) = |\{\sigma \in \Delta \mid c(\sigma) = A\}| \text{ and } h_A(\Delta) = \sum_{B \subset A} (-1)^{|A|-|B|} f_B(\Delta)$$

for any $A \subset [n]$. These refine $f(\Delta)$ and $h(\Delta)$ since $f_{k-1}(\Delta) = \sum_{|A|=k} f_A(\Delta)$ and $h_k(\Delta) = \sum_{|A|=k} h_A(\Delta)$.

Setting 3.1. In the rest of this section, when we say that Δ is an $(n-1)$-dimensional balanced simplicial complex on V, then we assume that V is a set of the form

$$V = \{y_{i,j} \mid 1 \le i \le n, 1 \le j \le d_i + 1\}$$

and the map $c : V \to [n]$ defined by $c(y_{i,j}) = i$ is a proper coloring of Δ.

We next explain an algebraic meaning of flag h-vectors. We refer the readers to [2, 12] for basic terms on commutative algebra such as the Cohen–Macaulay property, the Gorenstein property, system of parameters, etc. Let Δ be an $(n-1)$-dimensional balanced simplicial complex on V and $S = \Bbbk[V] = \Bbbk[y_{i,j} \mid 1 \le i \le n, 1 \le j \le d_i + 1]$. The **Stanley-Reisner ideal** of Δ is the ideal

$$I_\Delta = (y^\sigma \mid \sigma \subset V,\ \sigma \notin \Delta),$$

where $y^\sigma = \prod_{y_{i,j} \in \sigma} y_{i,j}$, and the **Stanley-Reisner ring** of Δ (over $\Bbbk$) is a quotient ring S/I_Δ. We note that the Krull dimension of S/I_Δ equals to $\dim \Delta + 1$ (see [12, II Theorem 1.3]). It is known that the balancedness guarantees that I_Δ and S/I_Δ are $\mathbb{Z}^n$-graded by the grading defined by $\deg(y_{i,j}) = \mathbf{e}_i \in \mathbb{Z}^n$. Let $\theta_1, \dots, \theta_n \in S$ be linear forms defined by

$$\theta_k = y_{k,1} + \cdots + y_{k,d_k+1} \quad (k = 1, 2, \dots, n)$$

(thus θ_k is the sum of the variables having degree $\mathbf{e}_k$). The following result is known.

Theorem 3.2 (Stanley (see [11] or [12, II §4]))**.** *With the same notation as above*

(i) $\theta_1, \dots, \theta_n$ *is a system of parameters of* S/I_Δ*, in particular,* $\dim_\Bbbk S/(I_\Delta + (\theta_1, \dots, \theta_n)) < \infty$.

(ii) $S/(I_\Delta + (\theta_1, \dots, \theta_n))$ *is a* $\mathbb{Z}^n$*-graded algebra concentrated in squarefree degrees, that is,*

$$S/(I_\Delta + (\theta_1, \dots, \theta_n)) = \bigoplus_{A \subset [n]} \big(S/(I_\Delta + (\theta_1, \dots, \theta_n))\big)_{\mathbf{e}_A}$$

(iii) *If* S/I_Δ *is Cohen-Macaulay then* $\dim_\Bbbk \big(S/(I_\Delta + (\theta_1, \dots, \theta_n))\big)_{\mathbf{e}_A} = h_A(\Delta)$ *for any* $A \subset [n]$.

(iv) *If* Δ *is a simplicial sphere then* $S/(I_\Delta + (\theta_1, \dots, \theta_n))$ *is an Artinian Gorenstein algebra of socle degree* $(1, 1, \dots, 1)$.

So if S/I_Δ is Cohen-Macaulay, then the flag h-vector of Δ is nothing but the $\mathbb{Z}^n$-graded Hilbert function of $S/(I_\Delta + (\theta_1, \dots, \theta_n))$. We call the above $\theta_1, \dots, \theta_n$ the **standard system of parameters** of S/I_Δ.

Now assume that Δ is an $(n-1)$-dimensional balanced simplicial sphere on V. By considering an isomorphism

$$\Bbbk[y_{i,j} \mid 1 \le i \le n, 1 \le j \le d_i+1]/(\theta_1,\dots,\theta_n) \cong \Bbbk[y_{i,j} \mid 1 \le i \le n, 1 \le j \le d_i],$$

we can see that there is an ideal J such that

$$\Bbbk[y_{i,j} \mid 1 \le i \le n, 1 \le j \le d_i]/J \cong \Bbbk[y_{i,j} \mid 1 \le i \le n, 1 \le j \le d_i+1]/(I_\Delta + (\theta_1,\dots,\theta_n)) \tag{3.1}$$

is an Artinian Gorenstein algebra of socle degree $(1,1,\dots,1) \in \mathbb{Z}^n$. Then from Theorem 3.2 and Lemma 2.4 we have the following corollary.

Corollary 3.3. *Let Δ be a balanced simplicial $(n-1)$-sphere on $V = \{y_{i,j} \mid 1 \le i \le n, 1 \le j \le d_i+1\}$ with the proper coloring $c : V \to [n]$ defined by $c(y_{i,j}) = i$, then*

$$h_A(\Delta) \le \min\{\prod_{i\in A} d_i, \prod_{i\in[n]\setminus A} d_i\} \quad \textit{for any } A \subset [n].$$

We say that a balanced simplicial $(n-1)$-sphere is a **balanced neighbourly simplicial sphere** (BNSS) of type $(d_1,\dots,d_n) \in \mathbb{Z}^n$ if

(i) $(h_{\{1\}}(\Delta),\dots,h_{\{n\}}(\Delta)) = (d_1,\dots,d_n)$, and
(ii) $h_A(\Delta) = \min\{\prod_{i\in A} d_i, \prod_{i\in[n]\setminus A} d_i\}$ for any $A \subset [n]$.

Theorem 3.2 and (3.1) tell that if a BNSS of type $(d_1,\dots,d_n)$ exists, then there is an Artinian Gorenstein algebra $R = \Bbbk[y_{i,j} \mid 1 \le i \le n, 1 \le j \le d_i]/J$ of socle degree $(1,1,\dots,1) \in \mathbb{Z}^n$ with $H(R,A) = \min\{\prod_{i\in A} d_i, \prod_{i\in[n]\setminus A} d_i\}$ for all $A \subset [n]$, and the polynomial $f \in \Bbbk[x_{i,j} \mid 1 \le i \le n, 1 \le j \le d_i]$ with $\operatorname{ann}(f) = J$ is a BNP of type $(d_1,\dots,d_n)$ (in particular, we can find such a polynomial f over any field). Below we explain how we can find such a polynomial from a combinatorial structure of a simplicial complex.

We recall simplicial chain complexes. Let Δ be an $(n-1)$-dimensional simplicial complex on $V = \{y_1,\dots,y_m\}$ with the total order $y_1 < \cdots < y_m$ and let

$$C_\bullet(\Delta,\Bbbk): \quad 0 \longrightarrow C_{n-1}(\Delta) \xrightarrow{\partial_{n-1}} \cdots \longrightarrow C_0(\Delta) \xrightarrow{\partial_0} C_{-1}(\Delta) \longrightarrow 0$$

be the reduced simplicial chain complex of Δ with coefficients in $\Bbbk$. Thus each $C_k(\Delta)$ is the $\Bbbk$-vector space with basis $\Delta_k = \{\sigma \in \Delta \mid \dim\sigma = k\}$ and the boundary map $\partial_k : C_k(\Delta) \to C_{k-1}(\Delta)$ is the $\Bbbk$-linear map such that for each $\sigma = \{y_{i_1},\dots,y_{i_k}\}$ with $i_1 < \cdots < i_k$ one has

$$\partial_k(\sigma) = \sum_{j=1}^{k} (-1)^{j-1} \sigma \setminus \{y_{i_j}\}.$$

It is a fundamental result in algebraic topology that if Δ is a simplicial $(n-1)$-sphere then there is a unique element (up to sign)

$$\mathfrak{c}_\Delta = \sum_{\sigma\in\Delta_{n-1}} \varepsilon_\sigma \sigma \in \operatorname{Ker}(\partial_{n-1})$$

with each $\varepsilon_\sigma \in \{\pm 1\}$. See e.g. [7, V Theorem 4.2]. We call $\mathfrak{c}_\Delta$ the **fundamental cycle** of Δ. We will consider fundamental cycles of balanced simplicial $(n-1)$-spheres on $V = \{y_{i,j} \mid$

$1 \le i \le n, 1 \le j \le d_i+1\}$, where we use a total order on V satisfying $y_{i,j} < y_{i',j'}$ whenever $i < i'$ (to define the boundary map of simplicial chain complexes). Also, for a balanced simplicial $(n-1)$-sphere on $V = \{y_{i,j} \mid 1 \le i \le n, 1 \le j \le d_i+1\}$ with the fundamental cycle $\mathfrak{c}_\Delta = \sum_{\sigma \in \Delta_{n-1}} \varepsilon_\sigma \sigma$, we define

$$\mathfrak{f}_\Delta = \sum_{\sigma \in \Delta_{n-1}} \varepsilon_\sigma x^\sigma \in \Bbbk[x_{i,j} \mid 1 \le i \le n, 1 \le j \le d_i+1].$$

The following fact is known to experts[3].

Proposition 3.4. *If Δ is a balanced simplicial $(n-1)$-sphere on V, then*

$$\operatorname{ann}(\mathfrak{f}_\Delta) = I_\Delta + (\theta_1, \dots, \theta_n).$$

Proof. We first prove that the LHS contains the RHS. By the definition of $\mathfrak{f}_\Delta$, the polynomial $\mathfrak{f}_\Delta$ does not contain any monomial which is divisible by some squarefree monomial x^σ with $\sigma \notin \Delta$, so it is clear that $I_\Delta \circ \mathfrak{f}_\Delta = 0$. We show that $\theta_i \circ \mathfrak{f}_\Delta = 0$ for any i. Since Δ is a simplicial $(n-1)$-sphere, for each $\tau \in \Delta_{n-2}$ there are exactly two vertices v_τ and v'_τ such that $\tau \cup \{v_\tau\}, \tau \cup \{v'_\tau\} \in \Delta_{n-1} = \{\sigma \in \Delta \mid \dim \sigma = n-1\}$. Also, the balancedness tells that $c(v_\tau) = c(v'_\tau)$, so

$$0 = \partial_{n-1}(\mathfrak{c}_\Delta) = \partial_{n-1}(\sum_{\sigma \in \Delta_{n-1}} \varepsilon_\sigma \sigma) = \sum_{\tau \in \Delta_{n-2}} (-1)^{c(v_\tau)-1}(\varepsilon_{\tau \cup \{v_\tau\}} + \varepsilon_{\tau \cup \{v'_\tau\}})\tau.$$

This tells that $\varepsilon_{\tau \cup \{v_\tau\}} + \varepsilon_{\tau \cup \{v'_\tau\}} = 0$ for any $\tau \in \Delta_{n-2}$. Using this, for each $i = 1, 2, \dots, n$ we have

$$\theta_i \circ \mathfrak{f}_\Delta = \sum_{\sigma \in \Delta_{n-1}} \varepsilon_\sigma x^\sigma = \sum_{\tau \in \Delta_{n-2},\ c(\tau) = [n] \setminus \{i\}} (\varepsilon_{\tau \cup \{v_\tau\}} + \varepsilon_{\tau \cup \{v'_\tau\}}) x^\tau = 0,$$

where we use $\varepsilon_{\tau \cup \{v_\tau\}} + \varepsilon_{\tau \cup \{v'_\tau\}} = 0$ for the last equality.

Now we proved that $\operatorname{ann}(\mathfrak{f}_\Delta) \supset I_\Delta + (\theta_1, \dots, \theta_n)$. Recall that both $S/\operatorname{ann}(\mathfrak{f}_\Delta)$ and $S/(I_\Delta + (\theta_1, \dots, \theta_n))$ are Artinian Gorenstein of socle degree $(1, 1, \dots, 1)$. Since, for Artinian Gorenstein algebras S/I and S/J having the same socle degree, $I \subset J$ is equivalent to $I = J$, it follows that $\operatorname{ann}(\mathfrak{f}_\Delta) = I_\Delta + (\theta_1, \dots, \theta_n)$. ☐

We call $\mathfrak{f}_\Delta$ the **standard inverse system** of Δ.

Example 3.5. Consider the 6-cycle Σ in Figure 1. This is a 1-dimensional balanced simplicial sphere and its fundamental cycle is

$$\mathfrak{c}_\Sigma = \{y_{1,1}, y_{2,1}\} - \{y_{1,2}, y_{2,1}\} + \{y_{1,2}, y_{2,2}\} - \{y_{1,3}, y_{2,2}\} + \{y_{1,3}, y_{2,3}\} - \{y_{1,1}, y_{2,3}\},$$

so the standard inverse system of Σ is

$$\mathfrak{f}_\Sigma = x_{1,1}x_{2,1} - x_{1,2}x_{2,1} + x_{1,2}x_{2,2} - x_{1,3}x_{2,2} + x_{1,3}x_{2,3} - x_{1,1}x_{2,3}.$$

[3]Indeed, the results in [6] provide a way to compute a polynomial f with $\operatorname{ann}^*(f) = I_\Delta + (\theta_1, \dots, \theta_d)$ for any simplicial $(n-1)$-sphere Δ and any linear system of parameters $\theta_1, \dots, \theta_n$.

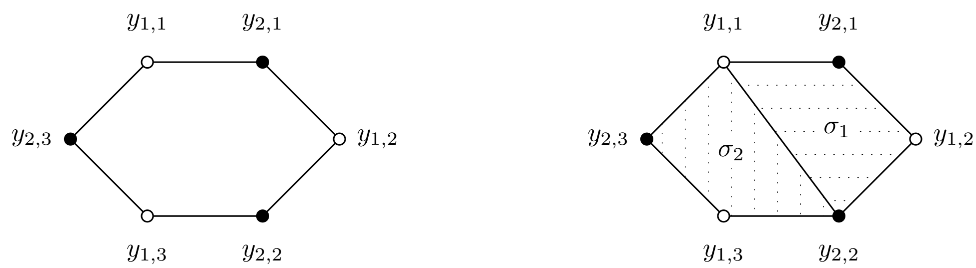

FIGURE 1. 6-cycle Σ and a cross-polytopal ball whose boundary is Σ.

One can easily check,

$$\begin{aligned}\operatorname{ann}(\mathfrak{f}_\Sigma) =&(y_{1,1}y_{1,2}, y_{1,1}y_{1,3}, y_{1,2}y_{1,3}, y_{2,1}y_{2,2}, y_{2,1}y_{2,3}, y_{2,2}y_{2,3}, y_{1,1}y_{2,2}, y_{1,2}y_{2,3}, y_{1,3}y_{2,1})\\ &+ (y_{1,1} + y_{1,2} + y_{1,3}, y_{2,1} + y_{2,2} + y_{2,3}),\end{aligned}$$

which equals to $I_\Sigma + (\theta_1, \theta_2)$.

Now let Δ be a balanced simplicial $(n-1)$-sphere on $V = \{y_{i,j} \mid 1 \le i \le n, 1 \le j \le d_i+1\}$. The standard inverse system $\mathfrak{f}_\Delta$ is actually redundant in the sense that $\operatorname{ann}(\mathfrak{f}_\Delta)$ contains linear forms $\theta_1, \dots, \theta_n$. But from $\mathfrak{f}_\Delta$ we can make another polynomial $\tilde{f}_\Delta$ such that $\operatorname{ann}(\tilde{\mathfrak{f}}_\Delta)$ does not contain any linear form as follows: Let $\tilde{x}_{i,j} = x_{i,j+1} - x_{i,j}$ for $1 \le i \le n$ and $1 \le j \le d_i$. Then the fact that

$$\theta_i \circ \mathfrak{f}_\Delta = 0$$

for $i = 1, 2, \dots, n$ tells that $\mathfrak{f}_\Delta$ is actually a polynomial in $\tilde{x}_{i,j}$. We call the polynomial $\tilde{\mathfrak{f}}_\Delta(\tilde{x}_{i,j})$ in variables $\tilde{x}_{i,j}$ with

$$\tilde{\mathfrak{f}}_\Delta(\tilde{x}_{i,j}) = \mathfrak{f}_\Delta(x_{i,j})$$

the **reduced inverse system** of Δ. Let $\tilde{S} = \Bbbk[\tilde{y}_{i,j} \mid 1 \le i \le n, 1 \le j \le d_i]$ be a polynomial ring with new variables $\tilde{y}_{i,j}$. Then we have

$$S/\operatorname{ann}(\mathfrak{f}_\Delta) \cong \tilde{S}/\operatorname{ann}(\tilde{\mathfrak{f}}_\Delta),$$

where we consider that $\operatorname{ann}(\tilde{\mathfrak{f}}_\Delta)$ is an ideal of $\tilde{S}$, and Theorem 3.2 and Proposition 3.4 tell that if Δ is a BNSS then $\tilde{\mathfrak{f}}_\Delta$ is a BNP, more precisely, we have the following statement.

Corollary 3.6. *If Δ is a balanced neighbourly simplicial sphere of type $(d_1, \dots, d_n)$ then the polynomial $\tilde{\mathfrak{f}}_\Delta$ is a balanced neighbourly polynomial of type $(d_1, \dots, d_n)$.*

We call $\tilde{\mathfrak{f}}_\Delta(\tilde{x}_{i,j})$ the **reduced inverse system** of Δ.

Example 3.7. In the previous example, we see

$$\begin{aligned}\mathfrak{f}_\Sigma &= x_{1,1}x_{2,1} - x_{1,2}x_{2,1} + x_{1,2}x_{2,2} - x_{1,3}x_{2,2} + x_{1,3}x_{2,3} - x_{1,1}x_{2,3}\\ &= (x_{1,1} - x_{1,2})(x_{2,1} - x_{2,2}) + (x_{1,1} - x_{1,3})(x_{2,2} - x_{2,3})\\ &= \tilde{x}_{1,1}\tilde{x}_{2,1} + (\tilde{x}_{1,1} + \tilde{x}_{1,2})\tilde{x}_{2,2},\end{aligned}$$

so

$$\tilde{\mathfrak{f}}_\Sigma = \tilde{x}_{1,1}\tilde{x}_{2,1} + (\tilde{x}_{1,1} + \tilde{x}_{1,2})\tilde{x}_{2,2}$$

is the reduced inverse system of Σ.

3.2. **Reduced inverse systems and cross-polytopal complexes.** Here we discuss a connection between reduced inverse systems and cross-polytopal complexes discussed in [5]. We refer the readers to [9, 10] for basics on combinatorial topology. For simplicial complexes Δ and Γ with disjoint vertices, the simplicial complex

$$\Delta * \Gamma = \{\sigma \cup \tau \mid \sigma \in \Delta,\ \tau \in \Gamma\}$$

is called the **join** of Δ and Γ. We write

$$C^{x_1,x_2,\dots,x_n}_{y_1,y_2,\dots,y_n} = \{x_1, y_1\} * \{x_2, y_2\} * \cdots * \{x_n, y_n\}, \tag{3.2}$$

where we consider that each $\{x_i, y_i\}$ is the 0-dimensional simplicial complex consisting of only two vertices x_i and y_i. This simplicial complex is the boundary complex of an n-dimensional cross-polytope.

An n-dimensional regular CW-complex is called a **cross-polytopal complex** if it is pure (every maximal cell have the same dimension) and the boundary of each n-dimensional cell is combinatorially isomorphic to the boundary complex of the n-dimensional cross-polytope (that is, the simplicial complex (3.2)). A **cross-polytopal n-ball** is a cross-polytopal complex which is homeomorphic to an n-dimensional ball.

We will consider cross-polytopal n-ball with the vertex set $\{y_{i,j} \mid 1 \le i \le n, 1 \le j \le d_i+1\}$ such that each maximal cell has the boundary of the form

$$\{y_{1,i_1}, y_{1,j_1}\} * \cdots * \{y_{n,i_n}, y_{n,j_n}\}, \tag{3.3}$$

where $i_1 < j_1, \dots, i_n < j_n$. We denote by $C^{i_1,\dots,i_n}_{j_1,\dots,j_n}$ the simplicial complex (3.3). Note that the boundary of such a cross-polytopal n-ball is a balanced simplicial $(n-1)$-sphere by the coloring defined by $c(y_{i,j}) = i$. An advantage of considering such a complex is that, as we will explain soon, the reduced inverse system of its boundary can be computed from the complex. To explain this, we first note that the fundamental cycle and (reduced) inverse system of (3.3) are easy to describe. Indeed, if $\partial\sigma$ equals to (3.3) then

$$\mathfrak{c}_{\partial\sigma} = \sum_{(k_1,\dots,k_n)\in\{i_1,j_1\}\times\cdots\times\{i_n,j_n\}} \varepsilon_{\{y_{1,k_1},\dots,y_{n,k_n}\}}\{y_{1,k_1}, \dots, y_{n,k_n}\}, \tag{3.4}$$

where $\varepsilon_{\{y_{1,k_1},\dots,y_{n,k_n}\}} = (-1)^{|\{\ell\in[n] \mid k_\ell = j_\ell\}|}$, and

$$\begin{aligned} \mathfrak{f}_{\partial\sigma} &= (x_{1,i_1} - x_{1,j_1}) \times \cdots \times (x_{n,i_n} - x_{n,j_n}) \\ &= (\tilde{x}_{1,i_1} + \cdots + \tilde{x}_{1,j_1-1}) \times \cdots \times (\tilde{x}_{n,i_n} + \cdots + \tilde{x}_{n,j_n-1}). \end{aligned} \tag{3.5}$$

Let $\mathcal{X}$ be a cross-polytopal n-ball and let $\mathcal{X}_n$ be the set of facets of $\mathcal{X}$. It is a standard fact in algebraic topology that there is a unique assignment $\sigma \in \mathcal{X}_n \to \varepsilon_\sigma \in \{\pm 1\}$ (up to sign) such that

$$\sum_{\sigma\in\mathcal{X}_n} \varepsilon_\sigma \mathfrak{c}_{\partial\sigma} \in C_{n-1}(\partial\mathcal{X}, \Bbbk) \quad \text{and} \quad \partial_{n-1}\Big(\sum_{\sigma\in\mathcal{X}_n} \varepsilon_\sigma \mathfrak{c}_{\partial\sigma}\Big) = 0,$$

where each $\mathfrak{c}_{\partial\sigma}$ is a fundamental cycle of $\partial\sigma$ and $C_{n-1}(\partial\mathcal{X}, \Bbbk)$ is the $(n-1)$th component of the simplicial homology of the boundary $\partial\mathcal{X}$ of $\mathcal{X}$. (Indeed, such an element is the image of a generator of $H_n(\mathcal{X}, \partial\mathcal{X})$ by the connecting homomorphism $H_n(\mathcal{X}, \partial\mathcal{X}) \to H_{n-1}(\partial\mathcal{X})$.) We call such an assignment an **orientation** of $\mathcal{X}$. Since the standard inverse system and the fundamental cycle of the simplicial complex (3.3) are naturally identified, we have the following statement.

Proposition 3.8. *Let $\mathcal{X}$ be a cross-polytopal n-ball with facets $\sigma_1, \dots, \sigma_m$ such that each $\partial\sigma_k$ is a simplicial complex of the form (3.3) and let $\sigma \in \mathcal{X}_n \to \varepsilon_\sigma \in \{\pm 1\}$ be an orientation of $\mathcal{X}$. Then $\partial\mathcal{X}$ is a balanced simplicial $(n-1)$-sphere with the standard inverse system*

$$\mathfrak{f}_{\partial\mathcal{X}} = \sum_{i=1}^{m} \varepsilon_{\sigma_i} \mathfrak{f}_{\partial\sigma_i}.$$

We give a quick example. Consider the 6-cycle Σ in Example 3.5. As shown in Figure 1, Σ is the boundary of a cross-polytopal ball with two facets σ_1, σ_2 such that $\partial\sigma_1 = C^{1,1}_{2,2}$ and $\partial\sigma_2 = C^{1,2}_{3,3}$. Then

$$\mathfrak{f}_{\partial\sigma_1} = (x_{1,1} - x_{1,2})(x_{2,1} - x_{2,2}) = \tilde{x}_{1,1}\tilde{x}_{2,1}$$

and

$$\mathfrak{f}_{\partial\sigma_2} = (x_{1,1} - x_{1,3})(x_{2,2} - x_{2,3}) = (\tilde{x}_{1,1} + \tilde{x}_{1,2})\tilde{x}_{2,2}.$$

Also, the orientation of $\mathcal{X}$ is $(\varepsilon_{\sigma_1}, \varepsilon_{\sigma_2}) = (+1, +1)$ (or $(-1, -1)$), and by the previous proposition we get

$$\tilde{\mathfrak{f}}_{\partial\mathcal{X}} = \tilde{\mathfrak{f}}_{\partial\sigma_1} + \tilde{\mathfrak{f}}_{\partial\sigma_2} = \tilde{x}_{1,1}\tilde{x}_{2,1} + (\tilde{x}_{1,1} + \tilde{x}_{1,2})\tilde{x}_{2,2}.$$

It often happens that a balanced simplicial sphere is the boundary of sch a cross-polytopal ball. See [4, 5] for more related results. Proposition 3.8 gives a convenient way to compute reduced inverse system of such balanced simplicial spheres.

3.3. **Existence problem of BNSSs and BNPs.** Let Δ be a balanced simplicial $(n-1)$-sphere. Recall that $f_{\{i\}}(\Delta) = h_{\{i\}} + 1$ equals to the number of vertices of Δ having color i. We call $(f_{\{1\}}(\Delta), \dots, f_{\{n\}}(\Delta))$ the **color class** of Δ. The main question on BNSSs is the following.

Problem 1. *For which $(d_1, \dots, d_n) \in \mathbb{N}^n$, a BNSS of type $(d_1, \dots, d_n)$ exists?*

This problem is interesting since if a BNSS exists, then by Corollary 3.3 it must have the largest flag h-vector among all balanced simplicial spheres having the same color class. The most interesting case of Problem 1 would be the case when $d_1, \dots, d_n$ are equal or more generally the case when $d_1 \le \cdots \le d_n \le d_1 + 1$ because a BNSS of such type has the largest h-vector among all balanced simplicial $(n-1)$-spheres having the same number of vertices as the following lemma shows.

Lemma 3.9. *Let $(d_1, \dots, d_n) \in \mathbb{N}^n$ with $d_1 \le \cdots \le d_n \le d_1 + 1$ and $v = d_1 + \cdots + d_n + n$. If Δ is a BNSS of type $(d_1, \dots, d_n)$ then for any balanced simplicial $(n-1)$-sphere Γ having v vertices one has $h_i(\Delta) \ge h_i(\Gamma)$ for all i.*

Proof. By Corollary 3.3 it suffices to show that if $(d_1,\dots,d_n)$ satisfies $d_1 \le \cdots \le d_n$ and $d_n > d_1+1$ then $(d'_1,\dots,d'_n) = (d_1+1, d_2,\dots,d_{n-1}, d_n-1)$ satisfies

$$\sum_{\{i_1,\dots,i_k\}\subset[n]} d_{i_1}\times\cdots\times d_{i_k} \le \sum_{\{i_1,\dots,i_k\}\subset[n]} d'_{i_1}\times\cdots\times d'_{i_k}$$

for any integer $k \ge 0$. The above inequality follows from the fact that the LHS in the above equation can be written as a sum of the following three terms

$$\sum_{1<i_1<\cdots<i_{k-1}<n} (d_1+d_n)\times d_{i_1}\times\cdots\times d_{i_{k-1}},$$
$$\sum_{1<i_1<\cdots<i_{k-2}<n} (d_1\times d_n)\times d_{i_1}\times\cdots\times d_{i_{k-2}},$$
$$\sum_{1<i_1<\cdots<i_k<n} d_{i_1}\times\cdots\times d_{i_k}$$

together with the fact that all non-negative integers $a<b$ satisfy $ab \le (a+1)(b-1)$. ☐

Problem 1 looks to be a hard problem in general. The boundary of a cross-polytope gives a BNSS of type $(1,1,\dots,1)$. Zheng [14] showed that BNSSs of type $(2,2,2,2)$ do not exist and BNSSs of types $(3,3,3,3)$ and type $(2,2,2,2,2)$ exist. As we already saw, if a BNSS of type $\boldsymbol{d}$ exits, then a BNP of type $\boldsymbol{d}$ exists over any field. This fact could be useful to study Problem 1. Indeed, it was proved by the second author [13] that BNPs of type $(2,2,2,2)$ do not exist over $\mathbb{Z}/2\mathbb{Z}$, which gives an alternative proof for the non-existence of BNSSs of type $(2,2,2,2)$ in [14]. Considering this, we ask

Problem 2. *For which $(d_1,\dots,d_n)\in\mathbb{N}^n$, a BNP of type $(d_1,\dots,d_n)$ exists (over any field)?*

It was proved in [13] that a BNP of type (k,k,k,k) exits over any field whenever $k>2$, but the existence of BNPs for other cases is wide open.

Remark 3.10. We give one remark about Problem 2. If the base field $\Bbbk$ is infinite, then a degree $(1,1,\dots,1)$ polynomial $f\in S$ with generic coefficients is a strong candidate of a BNP. However, we do not consider such polynomials with generic coefficients in this paper because if we consider an application to Problem 1 it is more important to consider polynomials with integer coefficients which is a BNP over $\mathbb{Z}/p\mathbb{Z}$ for all prime number p and find concrete ways to construct such BNPs.

We finally remark that not every $\mathbb{Z}^n$-graded Hilbert function of an Artinian Gorenstein algebra of socle degree $(1,1,\dots,1)$ can be realize as the flag h-vector of a balanced simplicial sphere. We see in Example 2.8 that, for every (d_1,d_2,d_3) with $d_1\le d_2\le d_3\le d_1\times d_2$, there is an Artinian Gorenstein $\Bbbk$-algebra R of socle degree $(1,1,1)$ with $H(R,\{1\})=d_1$, $H(R,\{2\})=d_2$ and $H(R,\{3\})=d_3$. The next statement shows that, for Stanley-Reisner rings, we have a stronger restrictions,

Proposition 3.11. *If Δ is a balanced simplicial 2-sphere, then $h_{\{3\}}(\Delta)\le h_{\{1\}}(\Delta)+h_{\{2\}}(\Delta)-1$.*

Proof. Let Δ be a balanced simplicial 2-sphere and let $h_i = h_{\{i\}}(\Delta)$ for $i = 1, 2, 3$. Let $v_1, \dots, v_{h_3+1}$ be the vertices of Δ having color 3. Consider the CW-complex $\mathcal{X}$ with maximal cells $F_1, \dots, F_{h_3+1}$ such that each F_i is a union of triangles of Δ containing v_i. Then the CW-complex $\mathcal{X}$ has h_3+1 facets, $f_{\{1,2\}}(\Delta)$ edges and $f_{\{1\}}(\Delta)+f_{\{2\}}(\Delta) = h_1+h_2+2$ vertices, so the Euler relation tells

$$(h_3 + 1) - f_{\{1,2\}}(\Delta) + (h_1 + h_2 + 2) = 2.$$

On the other hand, since each facet of $\mathcal{X}$ has at least 4 edges and since each edge of $\mathcal{X}$ is contained in exactly two facets of $\mathcal{X}$, we have

$$2 \times f_{\{1,2\}}(\Delta) \geq 4(h_3 + 1).$$

Substituting the Euler relation to the above inequality we get

$$2(h_1 + h_2 + h_3 + 1) \geq 4(h_3 + 1)$$

proving the desired inequality. ∎

4. Minimal BNPs

It is not easy to check whether a BNP of a fixed type exists or not. In this section we introduce minimal and weakly minimal BNPs, and consider methods to study their existence. Throughout this section, we use the following notation.

- $\boldsymbol{d} = (d_1, \dots, d_n) \in \mathbb{N}^n$ with $d_1 \leq \cdots \leq d_n$.
- $S = \Bbbk[x_{i,j} \mid 1 \leq i \leq n,\ 1 \leq j \leq d_i]$.
- $q = \lfloor \frac{n}{2} \rfloor$, where $\lfloor m \rfloor$ is the largest integer that does not exceed $m \in \mathbb{Q}$.
- $m_{\boldsymbol{d}} = \max_{A \subset [n]} \min\{\prod_{i \in A} d_i, \prod_{i \in [n] \setminus A} d_i\}$.
- $\mathrm{Mon}(S, A) =$ the set of monomials in S of degree $\mathbf{e}_A$ ($A \subset [n]$).
- $\mathrm{Mon}(f)$ = the set of monomials that appears in a polynomial $f \in S$, that is, if $f = \sum_{k=1}^{m} c_k \alpha_k$ where $c_k \in \Bbbk$ and α_k is a monomial, then $\mathrm{Mon}(f) = \{\alpha_1, \dots, \alpha_m\}$.
- For a squarefree monomial $\alpha = x_{1,i_1} \cdots x_{n,i_n}$ and a subset $A \subset [n]$,
$$\alpha_A = \prod_{k \in A} x_{k,i_k}.$$
- For a polynomial $f \in S$ of degree $(1, \dots, 1)$ and $A \subset [n]$,
$$\mathrm{Mon}(f, A) = \{\alpha_A \mid \alpha \in \mathrm{Mon}(f)\}.$$
- $\mathrm{shadow}(f, A) = \{\partial_\alpha f \mid \alpha \in \mathrm{Mon}(S, [n] \setminus A)\}$.

4.1. **Minimal BNPs.** Let $f \in S$ be a polynomial of degree $(1, 1, \dots, 1)$. For any $A \subset [n]$, we clearly have

$$H(f, A) \leq |\mathrm{shadow}(f, A)| \leq |\mathrm{Mon}(f, A)| \leq |\mathrm{Mon}(f)|,$$

where the second inequality follows since every element in $\mathrm{shadow}(f, A)$ can be written as a linear combination of elements in $\mathrm{Mon}(f, A)$, so if f is a BNP of type $\boldsymbol{d}$ then we must have

$\mathrm{Mon}(f) \geq m_{\boldsymbol{d}}$. Motivated by this fact, we say that a BNP f of type $\boldsymbol{d}$ is a **minimal BNP** if $|\mathrm{Mon}(f)| = m_{\boldsymbol{d}}$. For example, BNPs in Examples 2.7 and 2.8 are minimal BNPs.

Our first goal is to give a simple combinatorial criterion of minimal BNPs of type $(d, \dots, d)$. We first observe the following fact.

Lemma 4.1. *Suppose* $\boldsymbol{d} = (d, \dots, d)$*. If* $f \in S$ *is a BNP of type* $\boldsymbol{d}$ *then, for any* $A \subset [n]$ *with* $|A| = q$*, every monomial* $\alpha \in \mathrm{Mon}(S, A)$ *appears in some term of* f*, that is,* $\mathrm{Mon}(f, A) = \mathrm{Mon}(S, A)$*.*

Proof. Let $A \subset [n]$ with $|A| = q$. Since f is a BNP,

$$|\mathrm{shadow}(f, [n] \setminus A)| \geq H(f, [n] \setminus A) = m_{\boldsymbol{d}} = H(S, A) = |\mathrm{Mon}(S, A)|.$$

But this tells that $\partial_\alpha(f) \neq 0$ for any monomial $\alpha \in \mathrm{Mon}(S, A)$, proving the desired statement. □

The next theorem tells that the condition in Lemma 4.1 is actually sufficient to check whether a polynomial is a minimal BNP of type $(d, \dots, d)$ or not.

Theorem 4.2. *Suppose* $\boldsymbol{d} = (d, \dots, d)$*. Let* f *be a polynomial of degree* $(1, 1, \dots, 1)$ *with* $|\mathrm{Mon}(f)| = d^q$*. Then* f *is a minimal BNP of type* $\boldsymbol{d}$ *if and only if* $\mathrm{Mon}(f, A) = \mathrm{Mon}(S, A)$ *for any* $A \subset [n]$ *with* $|A| = q$*.*

Proof. The only if part follows from Lemma 4.1. We prove the if part. By Lemma 2.6 and (2.3), it suffices to show that $H(f, [n]/A) = d^q$ for any $A \subset [n]$ with $|A| = q$,

Recall $m_{\boldsymbol{d}} = d^q$ in this setting. Let $\mathrm{Mon}(f) = \{\alpha_1, \dots, \alpha_{d^q}\}$. The assumption tells that, for any $A \subset [n]$ with $|A| = q$,

$$|\mathrm{Mon}(f, A)| = |\mathrm{Mon}(S, A)| = d^q = |\mathrm{Mon}(f)|,$$

in particular, $(\alpha_1)_A, \dots, (\alpha_{d^q})_A$ are all different monomials and $|\mathrm{Mon}(f, B)| = d^q$ for any $B \subset [n]$ with $|B| \geq q$. This also means that $\partial_\alpha f$ is a monomial for every $\alpha \in \mathrm{Mon}(S, A)$ with $|A| = q$ and we have

$$\mathrm{shadow}(f, [n] \setminus A) = \{\partial_\alpha f \mid \alpha \in \mathrm{Mon}(S, A)\} = \{(\alpha_1)_{[n]\setminus A}, \dots, (\alpha_{d^q})_{[n]\setminus A}\} = \mathrm{Mon}(f, [n] \setminus A).$$

Since $|[n] \setminus A| \geq q$, the set $\mathrm{Mon}(f, [n] \setminus A)$ has cardinality $m_{\boldsymbol{d}} = d^q$, so the above equality tells $H(f, [n] \setminus A) = \dim_{\Bbbk} \mathrm{span}_{\Bbbk}(\mathrm{shadow}(f, [n] \setminus A)) = |\mathrm{Mon}(f, [n] \setminus A)| = d^q$ as desired. □

We note a few immediate corollaries of Theorem 4.2.

Corollary 4.3. *If* f *is a minimal BNP of type* $(d, \dots, d) \in \mathbb{N}^n$*, then* $\partial_\alpha f$ *is a monomial for any* $\alpha \in \mathrm{Mon}(S, A)$ *with* $|A| = q$*.*

Corollary 4.4. *If* f *is a minimal BNP of type* $(d, \dots, d)$ *with* $\mathrm{Mon}(f) = \{\alpha_1, \dots, \alpha_{d^q}\}$ *then the polynomial* $\alpha_1 + \cdots + \alpha_{d^q}$ *is also a minimal BNP.*

Note that Corollary 4.4 shows that the existence of a minimal BNP of type $(d, \dots, d)$ does not depend on a base field. We say that a minimal BNP f is **standard** if all the coefficients of terms in f equal to 1.

Example 4.5. Consider the polynomial

$$\begin{aligned} f = & x_1y_1z_3w_2 + x_1y_2z_1w_3 + x_1y_3z_2w_1 \\ & + x_2y_1z_2w_3 + x_2y_2z_3w_1 + x_2y_3z_1w_2 \\ & + x_3y_1z_1w_1 + x_3y_2z_2w_2 + x_3y_3z_3w_3. \end{aligned}$$

One can check that $\mathrm{Mon}(f,\{i,j\}) = \mathrm{Mon}(S,\{i,j\})$ for all $1 \le i < j \le 3$, so f is a minimal BNP of type $(3,3,3,3)$.

4.2. **A linear algebra method to create minimal BNPs.** Next we introduce a linear algebraic method to construct minimal BNPs. In this subsection, we regard $S = \Bbbk[x_{i,j} \mid 1 \le i \le n, 1 \le j \le d]$ as $S = \Bbbk[x_{i,j} \mid 1 \le i \le n, j \in \mathbb{Z}/d\mathbb{Z}]$. It was shown in [13, Theorem 3.2] that the polynomial

$$f = \sum_{1 \le i,j \le d} x_iy_jz_{j-i}w_{i+j} \in \Bbbk[x_p, y_q, z_r, w_s \mid p,q,r,s \in \mathbb{Z}/d\mathbb{Z}] \tag{4.1}$$

is a BNP of type (d,d,d,d) when d is odd. We extend this construction as follows.

Theorem 4.6. *Let $n > 0$ and $d > 0$ be positive integers, $S = \Bbbk[x_{i,j} \mid 1 \le i \le n, j \in \mathbb{Z}/d\mathbb{Z}]$, and $X = (a_{i,j})$ an $q \times n$ integer matrix. Then the polynomial*

$$g_X = \sum_{1 \le k_1,\dots,k_q \le d} \left(\prod_{j=1}^{n} x_{j,a_{1,j}k_1+\cdots+a_{q,j}k_q} \right)$$

is a BNP if and only if all maximal minors of X are non-zero and coprime to d.

Proof. Let $A \subset [n]$ with $|A| = q$. Then

$$\mathrm{Mon}(g_X, A) = \{\prod_{j \in A} x_{j,a_{1,j}k_1+\cdots+a_{q,j}k_q} \mid 1 \le k_1,\dots,k_q \le d\}.$$

Thus $\mathrm{Mon}(g_X, A) = \mathrm{Mon}(S, A)$ if and only if the linear map

$$(k_1,\dots,k_q) \to (k_1,\dots,k_q)X_A$$

gives a bijection from $(\mathbb{Z}/d\mathbb{Z})^q$ to itself, where X_A is the $q \times q$ submatrix of X whose columns corresponds to entries of A. The latter condition is equivalent to saying that $\det(X_A)$ is invertible in $\mathbb{Z}/d\mathbb{Z}$. This fact together with Theorem 4.2 guarantee that g_X is a minimal BNP if and only if all maximal minors of X are non-zero and coprime to d. □

We call a minimal BNP obtained by Theorem 4.6 a **linear minimal BNP**.

Example 4.7. Set $X = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \end{pmatrix}$. Then

$$g_X = \sum_{1 \le i,j \le d} x_iy_jz_{i-j}q_{i+j} \in \Bbbk[x_p, y_q, z_r, w_s \mid p,q,r,s \in \mathbb{Z}/d\mathbb{Z}]$$

is nothing but the polynomial (4.1).

Example 4.8. The matrix

$$X = \begin{pmatrix} 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & -1 & 1 & 1 \end{pmatrix}$$

has maximal minors $\pm 1, \pm 2, 3$. So g_X is a minimal BNP of type (d, d, d, d, d) when d is not divisible by 2 and 3.

Example 4.9. The matrix

$$X = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 & -1 & 1 \end{pmatrix}$$

has maximal minors $\pm 1, \pm 2, \pm 3$. So g_X is a BNP of type (d, d, d, d, d, d) when d is not divisible by 2 and 3.

If we take a matrix X in Theorem 4.6 generically then all the minors of X are non-zero. Hence we have the following corollary of the theorem.

Corollary 4.10. *Fix an integer $n > 0$. There are infinitely many prime numbers p such that a minimal BNP of type $(p, \dots, p) \in \mathbb{N}^n$ exists.*

The above corollary guarantees that a minimal BNP of type $(d, \dots, d)$ exists if d is a sufficiently large prime number. On the other hand, minimal BNPs of type $(d, \dots, d)$ do not exist when d is very small as the following statement shows.

Proposition 4.11. *A minimal BNP of type $(d, \dots, d) \in \mathbb{N}^n$ with $n \geq 4$ do not exist if $2 \leq d \leq \frac{n+1}{2}$.*

Proof. Let f be a standard minimal BNP of type $(d, d, \dots, d) \in \mathbb{N}^n$ with $d \geq 2$. We prove $d \geq \frac{n}{2} + 1$.

Since every monomial of degree $\mathbf{e}_{\{1,2,\dots,q\}}$ appears in some terms of f by Theorem 4.2, $\mathrm{Mon}(f)$ contains monomials of the form

$$x_{1,1} \cdots x_{q-1,1} x_{q,k} x_{q+1,b_{k,q+1}} \cdots x_{n,b_{k,n}}$$

for $k = 1, 2, \dots, d$. Also, we have $b_{k,i} \neq b_{\ell,i}$ for $k \neq \ell$ (since if $b_{k,i} = b_{\ell,i}$ then $\mathrm{Mon}(f)$ contains two monomials which are divisible by $x_{1,1} \cdots x_{q-1,1} x_{k,b_{k,i}}$, contradicting Corollary 4.3). Then by an appropriate permutation of variables $x_{i,j}$ with $i \geq q+1$, we may assume $b_{k,q+1} = \cdots = b_{k,n} = k$ for each k, so f is a polynomial of the form

$$\begin{aligned} f = & x_{1,1} \cdots x_{q-1,1} x_{q,1} x_{q+1,1} \cdots x_{n,1} \\ & + x_{1,1} \cdots x_{q-1,1} x_{q,2} x_{q+1,2} \cdots x_{n,2} \\ & \vdots \\ & + x_{1,1} \cdots x_{q-1,1} x_{q,d} x_{q+1,d} \cdots x_{n,d} \\ & + \cdots . \end{aligned}$$

Also, f must contain a term of the form

$$\alpha = x_{1,2} x_{2,1} \cdots x_{q,1} x_{q+1,a_{q+1}} \cdots x_{n,a_n}.$$

The numbers $a_{q+1},\dots,a_n$ must satisfy

$$a_i \neq a_j \text{ for any } i \neq j \tag{4.2}$$

because if $a_i = a_j$ then $x_{2,1}\cdots x_{q-1,1}x_{i,a_i}x_{j,a_i}$ appears both in α and $x_{1,1}\cdots x_{q-1,1}x_{q,a_i}\cdots x_{n,a_i}$, contradicting Corollary 4.3. We also have

$$a_{q+1},\dots,a_n \in \{2,3,\dots,d\} \tag{4.3}$$

since if $a_i = 1$ then $x_{2,1}\cdots x_{q,1}x_{i,1}$appears both in α and $x_{1,1}\dots,x_{q,1}x_{q+1,1}\cdots x_{n,1}$, again contradicting Corollary 4.3. The conditions (4.2) and (4.3) tell $n-q \leq d-1$, which proves $d \geq \frac{n}{2}+1$. □

4.3. **Weakly minimal BNPs and minimal BNSSs.** The idea of minimality of BNPs works even if we replace monomials with products of linear forms. Indeed, if

$$f = \sum_{k=1}^{p} \ell_{1,k} \times \cdots \times \ell_{n,k}$$

is a BNP of type $(d_1,\dots,d_n)$, where each $\ell_{i,j}$ are linear form in $\Bbbk[x_{i,1},\dots,x_{i,d_i}]$, then every element in shadow(f,A) is a linear combination of at most p polynomials, so we have $H(f,A) \leq p$ for all $A \subset [n]$. We say that a BNP f of type $\boldsymbol{d} = (d_1,\dots,d_n)$ is an **weakly minimal BNP** if f can be written in the form

$$f = \sum_{k=1}^{m_{\boldsymbol{d}}} \ell_{1,k} \times \cdots \times \ell_{n,k}$$

for some linear forms $\ell_{i,j}$.

Example 4.12. Consider the polynomial

$$\begin{aligned} f =& x_1(y_1+y_2)z_1w_2 + x_1(y_1+y_2+y_3)(z_2+z_3)(w_1+w_2) + x_1(y_2+y_3)z_2w_3 \\ &+ x_2y_1z_1w_1 + x_2y_2z_2w_2 + x_2y_3z_3w_3 \\ &+ x_3(y_1+y_2)z_2w_1 + x_3(y_1+y_2+y_3)(z_1+z_2)(w_2+w_3) + x_3(y_2+y_3)z_3w_2. \end{aligned}$$

This polynomial is a BNP of type $(3,3,3,3)$ so is an weakly minimal BNP.

The following statement gives a criterion of weakly minimal BNPs of type $(d,d,\dots,d) \in \mathbb{N}^n$ similar to Theorem 4.2 when n is even.

Theorem 4.13. *Let f be a polynomial in $\Bbbk[x_{i,j} \mid 1 \leq i \leq n, 1 \leq j \leq d]$ with*

$$f = \sum_{k=1}^{d^q} \ell_{1,k} \times \cdots \times \ell_{n,k}$$

where each $\ell_{i,j}$ is a linear form of degree $\mathbf{e}_i$. The following conditions are equivalent.

(i) *f is a BNP.*

(ii) *For each $A \subset [n]$ with $|A| = q$, the polynomials*

$$\prod_{i\in A} \ell_{i,1},\dots,\prod_{i\in A} \ell_{i,d^q}$$

are linearly independent.

Proof. Let $A \subset [n]$ with $|A| = q$. Let $X_A = (a^A_{k,\alpha})$ be the $d^q \times d^q$ matrix whose rows are indexed by $k = 1, 2, \dots, d^q$ and whose columns are indexed by monomials of degree $\mathbf{e}_A$ and the (k, α)-the entry of X_A is the coefficient of α in $\prod_{i \in A} \ell_{i,k}$. Since each $\prod_{i \in A} \ell_{i,k}$ is a squarefree polynomial,

$$\partial_\alpha(\prod_{i \in A} \ell_{i,k}) = a^A_{k,\alpha}$$

for any k and $\alpha \in \mathrm{Mon}(S, A)$. Thus, for any $\alpha \in \mathrm{Mon}(S, A)$ one has

$$\partial_\alpha f = \textstyle\sum_{k=1}^{d^q} a^A_{k,\alpha}(\prod_{i \notin A} \ell_{i,k}). \tag{4.4}$$

Now we prove (i)⇒(ii). Assume that f is a BNP. Then the sequence

$$(\partial_\alpha f \mid \alpha \in \mathrm{Mon}(S, A))$$

must be a sequence of linearly independent polynomials. So by (4.4) the matrix $X_A = (a^A_{k,\alpha})$ is a non-singular matrix. By the construction of X_A, rows of X_A are linearly independent if and only if the polynomials $\prod_{i \in A} \ell_{i,1}, \dots, \prod_{i \in A} \ell_{i,d^q}$ are linearly independent.

Next, we prove (ii)⇒(i). Fix $A \subset [n]$ with $|A| = q$. We prove that the sequence

$$(\partial_\alpha f \mid \alpha \in \mathrm{Mon}(S, A))$$

is a sequence of linearly independent polynomials. The condition (ii) tells that the matrix X_A is non-singular. The condition (ii) also tells that, for any $B \subset [n]$ with $|B| \geq q$, the sequence

$$(\prod_{i \in B} \ell_{i,k} \mid k = 1, 2, \dots, d^q)$$

is a sequence of linearly independent polynomials. These facts and (4.4) tell that

$$(\partial_\alpha f \mid \alpha \in \mathrm{Mon}(S, A))$$

is a sequence of linearly independent polynomials, proving $H(f, A) = H(S, A)$ for any $A \subset [n]$ with $|A| = q$ as desired. □

Theorem 4.13 tells that if $f = \sum_{k=1}^{d^q} \ell_{1,k} \cdots \ell_{n,k}$ is a BNP and n is odd then the polynomial $\sum_{k=1}^{d^q} \ell_{1,k} \cdots \ell_{n-1,k}$ is also a BNP. Hence we get the following corollary.

Corollary 4.14. *Suppose n is odd. If an weakly minimal BNP of type $(d, d, \dots, d) \in \mathbb{N}^n$ exists then an weakly minimal BNP of type $(d, d, \dots, d) \in \mathbb{N}^{n-1}$ also exists.*

The definition of weakly minimal BNPs has a combinatorial motivation. If $\mathcal{X}$ is a cross-polytopal n-ball in Proposition 3.8 and if $\mathcal{X}$ has m facets, then $\tilde{\mathfrak{f}}_{\partial \mathcal{X}}$ is a sum of m products of linear forms, so if $\partial \mathcal{X}$ is a BNSS of type $\boldsymbol{d}$ then $\mathcal{X}$ has at least $m_{\boldsymbol{d}}$ facets. Analogous to the minimality of BNPs, we define that a BNSS Δ of type $\boldsymbol{d} = (d_1, \dots, d_n)$ is a minimal BNSS if there is a cross-polytopal ball $\mathcal{X}$ satisfying the condition of Proposition 3.8 such that

(i) $\partial \mathcal{X} = \Delta$, and
(ii) $\mathcal{X}$ has exactly $m_{\boldsymbol{d}}$ facets.

By Proposition 3.8, we get the following statement.

Corollary 4.15. *If Δ is a minimal BNSS of type $\boldsymbol{d}$ then $\tilde{\mathfrak{f}}_\Delta$ is an weakly minimal BNP of type $\boldsymbol{d}$.*

Example 4.16. Zheng constructed a BNSS of type $(3, 3, 3, 3)$ in [14]. This example of Zheng is actually a minimal BNSS. Indeed, if we consider a cross-polytopal complex $\mathcal{X}$ whose facets $\sigma_1, \ldots, \sigma_9$ are given by

$$\begin{aligned} &\partial\sigma_1 = C^{1,1,1,2}_{2,3,2,3}, \quad \partial\sigma_2 = C^{1,1,2,1}_{2,4,4,3}, \quad \partial\sigma_3 = C^{1,2,2,3}_{2,4,3,4}, \\ &\partial\sigma_4 = C^{2,1,1,1}_{3,2,2,2}, \quad \partial\sigma_5 = C^{2,2,2,2}_{3,3,3,3}, \quad \partial\sigma_6 = C^{2,3,3,3}_{3,4,4,4}, \\ &\partial\sigma_7 = C^{3,1,2,1}_{4,3,3,2}, \quad \partial\sigma_8 = C^{3,1,1,2}_{4,4,3,4}, \quad \partial\sigma_9 = C^{3,2,3,2}_{4,4,4,3}, \end{aligned}$$

where $C^{1,1,1,2}_{2,3,2,3}$ means $C^{x_1,y_1,z_1,w_2}_{x_2,y_3,z_2,w_3}$ (see (3.2)), then the boundary of $\mathcal{X}$ is nothing but the BNSS of type $(3, 3, 3, 3)$ given n [14] (we refer the readers for the verification of this fact). We also note that the polynomial f given in Example 4.12 is the reduced inverse system of this BNSS (one can verify this from Proposition 3.8).

5. BNPs of type $(d-1, d, d, d)$, $(d-1, d-1, d, d)$ and $(d-1, d-1, d-1, d)$.

Lemma 3.9 suggests that in addition to BNPs and BNSSs of type $(d, d, \ldots, d)$, it is also interesting to consider existences of BNPs and BNSSs of type $(d-1, \ldots, d-1, d, \ldots, d)$. In this section, we study existence of such BNPs when $n = 4$. To simplify notation, we identify $x_k = x_{1,k}, y_k = x_{2,k}, z_k = x_{3,k}, w_k = x_{4,k}$ throughout this section. Below is the main result of this section.

Theorem 5.1. *Let $f \in \Bbbk[x_p, y_q, z_r, w_s \mid 1 \le p, q, r, s \le d]$ be a standard minimal BNP of type (d, d, d, d) with $d \ge 3$.*

(i) *The polynomial $f_1 = f|_{x_d=0}$ is a BNP of type $(d-1, d, d, d)$.*
(ii) *The polynomial $f_2 = f_1|_{y_d=y_1+\cdots+y_{d-1}}$ is a BNP of type $(d-1, d-1, d, d)$.*
(iii) *If $x_d y_d z_d w_d \in \mathrm{Mon}(f)$ and $d \ge 4$, then $f_3 = f_2|_{z_d=z_1+\cdots+z_{d-1}}$ is a BNP of type $(d-1, d-1, d-1, d)$.*

Proof. We remark that f has the following properties by Theorem 4.2 and Corollary 4.3:

(5.1) $\quad \frac{\partial}{\partial x_{i,p}} \frac{\partial}{\partial x_{j,q}} f$ is a monomial for each $1 \le i < j \le 4$ and $1 \le p, q \le d$.

(5.2) $\quad \mathrm{Mon}(f, \{i, j\}) = \mathrm{shadow}(f, \{i, j\}) = \{x_{i,p} x_{j,q} \mid 1 \le p, q \le d\}$ for each $1 \le i < j \le 4$.

(i) By Lemma 2.6, what we must prove is

$$H(f_1, \{1, 2\}) = H(f_1, \{1, 3\}) = H(f_1, \{1, 4\}) = (d-1)d.$$

By the symmetry of colors $2, 3, 4$ it suffices to consider $H(f_1, \{1, 2\})$. The property (5.2) tells

$$\mathrm{shadow}(f_1, \{1, 2\}) = \{x_i y_j|_{x_d=0} \mid 1 \le i, j \le d\} = \{x_i y_j \mid 1 \le i \le d-1, 1 \le j \le d\}$$

has cardinality $(d-1)d$, which proves $H(f_1, \{1, 2\}) = (d-1)d$.

(ii) By Lemma 2.6, what we must prove is

$$H(f_2,\{1,2\})=(d-1)^2,\ \ H(f_2,\{1,3\})=H(f_2,\{1,4\})=(d-1)d.$$

We first consider $H(f_2,\{1,2\})$. Observe

$$\frac{\partial}{\partial z_i}\frac{\partial}{\partial w_j}f_2=\left(\frac{\partial}{\partial z_i}\frac{\partial}{\partial w_j}f\right)|_{x_d=0,y_d=y_1+\cdots+y_{d-1}}.$$

Then by (5.2)

$$\mathrm{shadow}(f_2,\{1,2\})=\{x_iy_j|_{x_d=0,y_d=y_1+\cdots+y_{d-1}}\mid 1\le i,j\le d\}$$

contains the set $\{x_iy_j\mid 1\le i,j\le d-1\}$, so we have

$$\mathrm{span}_{\Bbbk}(\mathrm{shadow}(f_2,\{1,2\}))=\mathrm{span}_{\Bbbk}\{x_iy_j\mid 1\le i,j\le d-1\},$$

proving $H(f_2,\{1,2\})=(d-1)^2$.

Next, we prove $H(f_2,\{1,3\})=H(f_2,\{1,4\})=(d-1)d$. Observe for each $1\le i\le d-1$ and $1\le j\le d$ one has

$$\frac{\partial}{\partial y_i}\frac{\partial}{\partial w_j}f_2=\frac{\partial}{\partial y_i}\frac{\partial}{\partial w_j}f_1+\frac{\partial}{\partial y_d}\frac{\partial}{\partial w_j}f_1=(\frac{\partial}{\partial y_i}\frac{\partial}{\partial w_j}f)|_{x_d=0}+(\frac{\partial}{\partial y_d}\frac{\partial}{\partial w_j}f)|_{x_d=0}. \tag{5.3}$$

We note that $\frac{\partial}{\partial y_i}\frac{\partial}{\partial w_j}f_2\ne 0$ since if it is zero then $\frac{\partial}{\partial y_i}\frac{\partial}{\partial w_j}f=x_dz_s$ and $\frac{\partial}{\partial y_d}\frac{\partial}{\partial w_j}f=x_dz_t$ for some s,t but this tells that $\frac{\partial}{\partial x_d}\frac{\partial}{\partial w_j}f$ is not a monomial, contradicting (5.1). Now, since $\frac{\partial}{\partial y_i}\frac{\partial}{\partial w_j}f$ ($1\le i,j\le d$) are distinct monomials, the formula (5.3) tells that $\frac{\partial}{\partial y_i}\frac{\partial}{\partial w_j}f_2$ ($1\le i\le d-1$, $1\le j\le d$) are linearly independent, so $H(f_2,\{1,3\})=(d-1)d$. By the symmetry of colors 3 and 4, we also have $H(f_2,\{1,4\})=(d-1)d$.

(iii) By Lemma 2.6, what we must prove is

$$H(f_3,\{1,2\})=H(f_3,\{1,3\})=H(f_3,\{2,3\})=(d-1)^2$$

and

$$H(f_3,\{4\})=d.$$

For each $1\le i\le d-1$ and $1\le j\le d$, we have

$$\begin{aligned}\frac{\partial}{\partial z_i}\frac{\partial}{\partial w_j}f_3&=\frac{\partial}{\partial z_i}\frac{\partial}{\partial w_j}f_2+\frac{\partial}{\partial z_d}\frac{\partial}{\partial w_j}f_2\\ &=(\frac{\partial}{\partial z_i}\frac{\partial}{\partial w_j}f)|_{x_d=0,y_d=y_1+\cdots+y_{d-1}}+(\frac{\partial}{\partial z_d}\frac{\partial}{\partial w_j}f)|_{x_d=0,y_d=y_1+\cdots+y_{d-1}}.\end{aligned} \tag{5.4}$$

By (5.2), one of $\frac{\partial}{\partial z_1}\frac{\partial}{\partial w_j}f,\dots,\frac{\partial}{\partial z_d}\frac{\partial}{\partial w_j}f$ is divisible by x_d (otherwise x_dw_j does not appear in $\mathrm{Mon}(f,\{1,4\})$), which tells

$$\left(\frac{\partial}{\partial z_d}\frac{\partial}{\partial w_j}f\right)|_{x_d=0,y_d=y_1+\cdots+y_{d-1}}=\frac{\partial}{\partial z_i}\frac{\partial}{\partial w_j}f_3$$

for some i unless $(\frac{\partial}{\partial z_d}\frac{\partial}{\partial w_j}f)|_{x_d=0}=0$. But this tells

$$\begin{aligned}&\mathrm{span}_{\Bbbk}\{\frac{\partial}{\partial z_i}\frac{\partial}{\partial w_j}f_3\mid 1\le i\le d-1,1\le j\le d\}\\ &=\mathrm{span}_{\Bbbk}\{(\frac{\partial}{\partial z_i}\frac{\partial}{\partial w_j}f)|_{x_d=0,y_d=y_1+\cdots+y_{d-1}}\mid 1\le i\le d-1,1\le j\le d\}\\ &\quad+\mathrm{span}_{\Bbbk}\{(\frac{\partial}{\partial z_d}\frac{\partial}{\partial w_j}f)|_{x_d=0,y_d=y_1+\cdots+y_{d-1}}\mid 1\le j\le d\}\\ &=\mathrm{span}_{\Bbbk}(\mathrm{shadow}(f_2,\{1,2\})),\end{aligned}$$

so $H(f_3,\{1,2\}) = H(f_2,\{1,2\}) = (d-1)^2$ by (ii). This also prove $H(f_3,\{1,3\}) = (d-1)^2$ by the symmetry of colors 2 and 3.

We next consider $H(f_3,\{2,3\})$. Let $\frac{\partial}{\partial x_d}\frac{\partial}{\partial w_k} f = y_{p_k} z_{q_k}$ for $k = 1,2,\dots,d$. Note that $\{p_1,\dots,p_d\} = \{q_1,\dots,q_d\} = \{1,2,\dots,d\}$ by (5.2). We have

$$\begin{aligned}\mathrm{shadow}(f_3,\{2,3\}) &= \{\tfrac{\partial}{\partial x_i}\tfrac{\partial}{\partial w_j} f_3 \mid 1 \le i \le d-1, 1 \le j \le d\}\\ &= \{y_i z_j\,|_{y_d=y_1+\cdots+y_{d-1}, z_d = z_1+\cdots+z_{d-1}} \mid (i,j) \notin \{(p_1,q_1),\dots,(p_d,q_d)\}\}.\end{aligned}$$

But since $p_1,\dots,p_d$ are distinct we have

$$\begin{aligned}&\mathrm{span}_{\Bbbk}\{y_i z_j|_{y_d=y_1+\cdots+y_{d-1}} \mid (i,j) \notin \{(p_1,q_1),\dots,(p_d,q_d)\}\}\\ &= \mathrm{span}_{\Bbbk}\{y_i z_j \mid 1 \le i \le d-1,\ 1 \le j \le d\}\end{aligned}$$

which implies

$$\begin{aligned}\mathrm{span}_{\Bbbk}(\mathrm{shadow}(f_3,\{2,3\})) &= \mathrm{span}_{\Bbbk}\{y_i z_j|_{z_d=z_1+\cdots+z_{d-1}} \mid 1 \le i \le d-1, 1 \le j \le d\}\\ &= \mathrm{span}_{\Bbbk}\{y_i z_j \mid 1 \le i,j \le d-1\},\end{aligned}$$

proving $H(f,\{2,3\}) = (d-1)^2$.

Finally we consider $H(f_3,\{4\})$. Fix $1 \le i \le 4$. We claim that $\frac{\partial}{\partial x_a}\frac{\partial}{\partial y_b}\frac{\partial}{\partial z_c} f_3 = w_i$ for some a,b,c. Let $\frac{\partial}{\partial w_i}\frac{\partial}{\partial x_k} f = y_{s_k} z_{t_k}$ for $k = 1,2,\dots,d$. Then $s_1,\dots,s_d$ are all different (otherwise, $\frac{\partial}{\partial w_i}\frac{\partial}{\partial y_\ell} f$ is not a monomial for some ℓ). Also, $t_1,\dots,t_d$ are all different. Since $r \ge 3$, there is an $r \in \{1,2,3\}$ such that $p_r \ne d$ and $q_r \ne d$. We claim that $\frac{\partial}{\partial x_r}\frac{\partial}{\partial y_{p_r}}\frac{\partial}{\partial z_{q_r}} f_3 = w_i$. (Note that $r \ne d$ since we are assuming $d > 3$.) Suppose contrary that $\frac{\partial}{\partial x_r}\frac{\partial}{\partial y_{p_r}}\frac{\partial}{\partial z_{q_r}} f_3 \ne w_i$. Then either $x_r y_d z_{q_r}$, $x_r y_{p_r} z_d$ or $x_r y_d z_d$ must appears in some term of f. The first two cases are impossible by (5.1) since $x_r y_{p_r}$ and $x_r z_{q_r}$ appears in $x_r w_i y_{p_r} z_{q_r}$. The last case is also impossible by (5.1) and our assumption that $x_d y_d z_d w_d \in \mathrm{Mon}(f)$. Now we proved that $w_1, w_2, \dots, w_d \in \mathrm{shadow}(f_3,\{4\})$, which implies $H(f,\{4\}) = d$. □

Example 5.2. Consider the minimal BNP

$$f = x_1y_1z_3w_2 + x_1y_2z_1w_3 + x_1y_3z_2w_1 + x_2y_1z_2w_3 + x_2y_2z_3w_1 + x_2y_3z_1w_2 + x_3y_1z_1w_1 + x_3y_2z_2w_2 + x_3y_3z_3w_3$$

given in Example 4.5. By Theorem 5.1,

$$f_1 = x_1y_1z_3w_2 + x_1y_2z_1w_3 + x_1y_3z_2w_1 + x_2y_1z_2w_3 + x_2y_2z_3w_1 + x_2y_3z_1w_2$$

is a BNP of type $(2,3,3,3)$ and

$$f_2 = x_1y_1z_3w_2 + x_1y_2z_1w_3 + x_1(y_1+y_2)z_2w_1 + x_2y_1z_2w_3 + x_2y_2z_3w_1 + x_2(y_1+y_2)z_1w_2$$

is a BNP of type $(2,2,3,3)$. Also, one can check that

$$f_3 = x_1y_1(z_1+z_2)w_2 + x_1y_2z_1w_3 + x_1(y_1+y_2)z_2w_1 + x_2y_1z_2w_3 + x_2y_2(z_1+z_2)w_1 + x_2(y_1+y_2)z_1w_2$$

is a BNP of type $(2,2,2,3)$ (although this case is not covered by Theorem 5.1(iii) since it assumes $d \ge 4$).

When d is odd or divisible by 4, a minimal BNP of type (d,d,d,d) was constructed in [13]. Using this result and Theorem 5.1, we get the following corollary.

Corollary 5.3. *Let $d > 1$ be odd or divisible by 4 and let $\boldsymbol{d} \in \{(d-1,d,d,d),(d-1,d-1,d,d),(d-1,d-1,d-1,d)\}$. There is a BNP of type $\boldsymbol{d}$.*

6. BNSSs of type $(2, 4k-1, 4k-1, 4k-1)$

In this section, we construct a minimal BNSS of type $(2, 4k-1, 4k-1, 4k-1)$. We first introduce some notation used in this section. In this section we consider that the vertices of complexes are $x_1, \dots, y_1, \dots, z_1, \dots, w_1, \dots$ with the coloring $c(x_i) = 1$, $c(y_i) = 2$, $c(z_i) = 3$, $c(w_i) = 4$. Also, for positive integers $a, b, c, d, a', b', c', d'$, we write

$$C^{a,b,c,d}_{a',b',c',d'} = \{x_a, x_{a'}\} * \{y_b, y_{b'}\} * \{z_c, z_{c'}\} * \{w_d, w_{d'}\}.$$

For a simplicial complex Δ and its faces $F_1, \dots, F_r$ we write

$$\Delta \setminus \{F_1, \dots, F_r\} = \{G \in \Delta \mid G \text{ does not contain any of } F_i\}.$$

Also, we write $\langle G_1, \dots, G_m \rangle$ for the simplicial complex generated by $G_1, \dots, G_m$, that is,

$$\langle G_1, \dots, G_m \rangle = \{G \mid G \subset G_i \text{ for some } i\}.$$

The main result in this section is the following theorem.

Theorem 6.1. *There is a minimal BNSS of type* $(2, 4k-1, 4k-1, 4k-1)$ *for any positive integer* k.

Proof. The idea of the proof is motivated from the cross-polytopal complex given in Example 4.16. Let $\mathcal{X}_1$ be the cross-polytopal complex with facets $\sigma_1, \dots, \sigma_{4k-1}$ such that

$$\partial \sigma_i = C^{2,i,i,i}_{3,i+1,i+1,i+1}.$$

Let $\mathcal{X}_2$ be the cross-polytopal complex with facets $\tau_{0,1}, \tau_{0,2}, \tau_{0,3}, \tau_{1,0}, \tau_{1,1}, \tau_{1,2}, \tau_{1,3}, \dots, \tau_{k-1,0}, \tau_{k-1,1}, \tau_{k-1,2}, \tau_{k-1,3}$ (we do not define $\tau_{0,0}$) such that

$$\partial \tau_{i,0} = C^{1,4i,4i-2,4i}_{2,4i+1,4i+1,4i+3},$$
$$\partial \tau_{i,1} = C^{1,4i+1,4i+1,4i+2}_{2,4i+3,4i+2,4i+3},$$
$$\partial \tau_{i,2} = C^{1,4i+1,4i+2,4i+1}_{2,4i+4,4i+4,4i+3},$$
$$\partial \tau_{i,3} = C^{1,4i+2,4i+2,4i+3}_{2,4i+4,4i+3,4i+4}.$$

We will prove that $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2$ is a cross-polytopal ball and $\partial \mathcal{X}$ is a BNSS of type $(2, 4k-1, 4k-1, 4k-1)$. We use the following known fact (see [1, §12])

(6.1) The union of two PL d-balls B_1, B_2 is again a PL d-ball if $B_1 \cap B_2$ is a PL $(d-1)$-ball belonging to the boundary of both B_1 and B_2.

Step1. We first prove that $\mathcal{X}_1$ is a cross-polytopal ball such that

$$\partial \mathcal{X}_1 = \partial \sigma_1 \cup \dots \cup \partial \sigma_{4k-1} \setminus \{\{y_q, z_q, w_q\} \mid q \in \{2, 3, \dots, 4k-1\}\}.$$

Let Γ_i be the subcomplex of $\mathcal{X}_1$ generated by $\sigma_1, \dots, \sigma_{i-1}$. Then by the definition of σ_i the complex $\Gamma_i \cap \partial \sigma_i$ is the simplicial complex generated by $\{x_2, y_i, z_i, w_i\}$ and $\{x_3, y_i, z_i, w_i\}$ for $i = 2, 3, \dots, 4k-1$. Then the desired statement follows from (6.1).

Step 2. We next prove that $\mathcal{X}_2$ is a cross-polytopal ball such that

$$\partial \mathcal{X}_2 = \Big(\bigcup_{(i,j)} \partial \tau_{i,j}\Big) \setminus E,$$

where (i,j) runs over elements in $\{(i,j) \mid 0 \le i \le k-1, 0 \le j \le 3\} \setminus \{(0,0)\}$ and E is the set

$$\left(\bigcup_{i=0}^{k-1} \left\{ \begin{array}{c} \{y_{4i}, z_{4i-2}, w_{4i}\}, \{y_{4i+1}, z_{4i+1}, w_{4i+3}\}, \\ \{y_{4i+1}, z_{4i+2}, w_{4i+3}\}, \{y_{4i+4}, z_{4i+2}, w_{4i+3}\} \end{array} \right\}\right) \setminus \big\{\{y_0, z_{-2}, w_0\}, \{y_{4k}, z_{4k-2}, w_{4k-1}\}\big\}.$$

Let $\Gamma_{i,j}$ be the subcomplex of $\mathcal{X}_2$ generated by all $\tau_{k,\ell}$ with $k < i$ and '$k = i$ and $\ell < j$'. The statement of Step 2 follows from (6.1) using the following routine computations:

$$\begin{aligned}
\Gamma_{i,0} \cap \partial\tau_{i,0} &= \langle \{x_1, y_{4i}, z_{4i-2}, w_{4i}\}, \{x_2, y_{4i}, z_{4i-2}, w_{4i}\} \rangle, \\
\Gamma_{i,1} \cap \partial\tau_{i,1} &= \langle \{x_1, y_{4i+1}, z_{4i+1}, w_{4i+3}\}, \{x_2, y_{4i+1}, z_{4i+1}, w_{4i+3}\} \rangle, \\
\Gamma_{i,2} \cap \partial\tau_{i,2} &= \langle \{x_1, y_{4i+1}, z_{4i+2}, w_{4i+3}\}, \{x_2, y_{4i+1}, z_{4i+2}, w_{4i+3}\} \rangle, \\
\Gamma_{i,3} \cap \partial\tau_{i,3} &= \langle \{x_1, y_{4i+4}, z_{4i+2}, w_{4i+3}\}, \{x_2, y_{4i+4}, z_{4i+2}, w_{4i+3}\} \rangle.
\end{aligned}$$

(For example, to see the first equality, one checks $\{x_1, y_{4i}, z_{4i-2}, w_{4i}\}$ and $\{x_2, y_{4i}, z_{4i-2}, w_{4i}\}$ are in $\Gamma_{i,0} \cap \partial\tau_{i,0}$ and $y_{4i+1}, z_{4i+1}, w_{4i+3} \notin \Gamma_{i,0}$.)

Step 3. We prove that $\mathcal{X}_1 \cap \mathcal{X}_2$ is a topological ball with no interior edges. Again, by routine computations, one has

$$\mathcal{X}_1 \cap \partial\tau_{i,0} = \langle \{x_2, y_{4i}, z_{4i+1}, w_{4i}\}, \{x_2, y_{4i+1}, z_{4i+1}, w_{4i}\} \rangle, \tag{6.2}$$

$$\mathcal{X}_1 \cap \partial\tau_{i,1} = \left\langle \begin{array}{c} \{x_2, y_{4i+1}, z_{4i+1}, w_{4i+2}\}, \{x_2, y_{4i+1}, z_{4i+2}, w_{4i+2}\}, \\ \{x_2, y_{4i+3}, z_{4i+2}, w_{4i+2}\}, \{x_2, y_{4i+3}, z_{4i+2}, w_{4i+3}\} \end{array} \right\rangle, \tag{6.3}$$

$$\mathcal{X}_1 \cap \partial\tau_{i,2} = \left\langle \{x_2, y_{4i+1}, z_{4i+2}, w_{4i+1}\}, \{x_2, y_{4i+4}, z_{4i+4}, w_{4i+3}\}, \{x_2, z_{4i+2}, w_{4i+3}\} \right\rangle, \tag{6.4}$$

$$\mathcal{X}_1 \cap \partial\tau_{i,3} = \left\langle \begin{array}{c} \{x_2, y_{4i+2}, z_{4i+2}, w_{4i+3}\}, \{x_2, y_{4i+2}, z_{4i+3}, w_{4i+3}\}, \\ \{x_2, y_{4i+4}, z_{4i+3}, w_{4i+3}\}, \{x_2, y_{4i+4}, z_{4i+3}, w_{4i+4}\} \end{array} \right\rangle. \tag{6.5}$$

In Figure 2, we show figures of the union of (6.3),(6.4),(6.5) as well as the complex (6.2), both of which are topologically balls (we delete the vertex x_2 so that the figure becomes 2-dimensional). It can be easily seen from these figures that the union of $\mathcal{X}_1 \cap \partial\tau_{i,j}$ for all i,j forms a simplicial ball with no interior edges (no interior vertices in the figure since we are deleting x_2 in the figure). Also, since none of the facets of complexes (6.2)–(6.5) contain an element in E, we have $\mathcal{X}_1 \cap \mathcal{X}_2 = \cup_{(i,j)}(\mathcal{X}_1 \cap \partial\tau_{i,j})$. These prove the desired statement.

Step 4. We prove the statement of the theorem. By Steps 1–3 and (6.1), the complex $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2$ is a cross-polytopal ball. Also, since $\mathcal{X}_1 \cap \mathcal{X}_2$ has no interior edges, the edge set of $\partial\mathcal{X}$ is the union of the edge sets of $\mathcal{X}_1$ and $\mathcal{X}_2$. One can see from Steps 1 and 2 that $\{x_i, y_j\}, \{x_i, z_j\}, \{x_i, w_j\}$ are edges of $\partial\mathcal{X}_1 \cup \partial\mathcal{X}_2$ for all i,j, so we have

- $f_{\{1,2\}}(\partial\mathcal{X}) = f_{\{1,3\}}(\partial\mathcal{X}) = f_{\{1,4\}}(\partial\mathcal{X}) = 3 \times 4k$,
- $f_{\{1\}}(\partial\mathcal{X}) = 3$, and
- $f_{\{2\}}(\partial\mathcal{X}) = f_{\{3\}}(\partial\mathcal{X}) = f_{\{4\}}(\partial\mathcal{X}) = 4k$.

But this means $h_{\{1,2\}}(\partial\mathcal{X}) = h_{\{1,3\}}(\partial\mathcal{X}) = h_{\{1,4\}}(\partial\mathcal{X}) = 2 \times (4k-1)$ by the definition of flag h-vectors, so $\partial\mathcal{X}$ is a BNSS. □

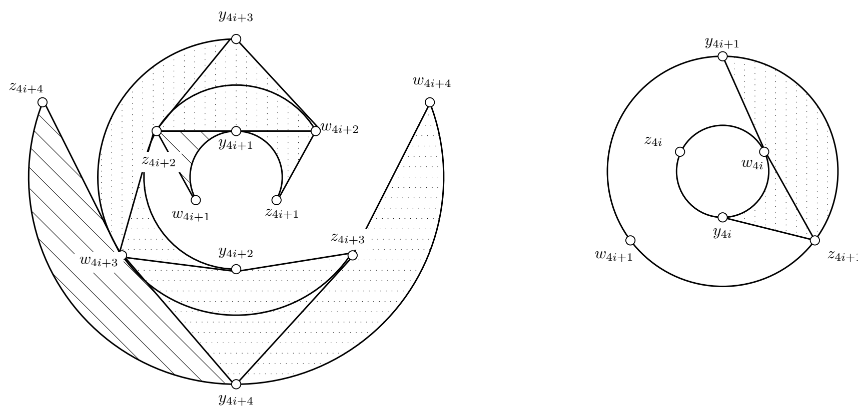

FIGURE 2. The left figure is the union of (6.3),(6.4) and (6.5). The right figure is the complex (6.2).

7. Remarks

In this last section, we discuss a few special cases of Problems 1 and 2.

Type $(2, 2, \ldots, 2)$**.** Probably first non-trivial and interesting special cases of Problems 1 and 2 are the cases $(d_1, \ldots, d_n) = (2, 2, \ldots, 2)$. Indeed, it is known that there are no BNSSs of type $(2, 2, 2, 2)$ [14] and it is even proved that there are no BNPs of type $(2, 2, 2, 2)$ over $\mathbb{Z}/2\mathbb{Z}$ [13].

There is a BNSS of type $(2, 2, 2, 2, 2)$ (see [14]). One may ask if this BNSS of type $(2, 2, 2, 2, 2)$ is a minimal BNSS. But there are no minimal BNSSs of type $(2, 2, 2, 2, 2)$ because the existence of such a minimal BNSS implies the existence of an weakly minimal BNP of type $(2, 2, 2, 2, 2)$ which also implies the existence of an weakly minimal BNP of type $(2, 2, 2, 2)$ by Corollary 4.14, but as we explained above there are no BNPs of type $(2, 2, 2, 2)$.

We do not know if there is a BNSS of type $(2, 2, 2, 2, 2, 2)$ but there is an weakly minimal BNP of type $(2, 2, 2, 2, 2, 2)$. Indeed we checked that the polynomial

$$\begin{aligned}
&x_1y_2(z_1+z_2)w_2(u_1+u_2)t_2 + x_1(y_1+y_2)z_2w_2u_1(t_1+t_2) + x_1y_1z_2w_1u_2t_2\\
&+ x_2y_1(z_1+z_2)w_1(u_1+u_2)t_1 + x_2(y_1+y_2)z_1w_1u_2(t_1+t_2) + x_2y_2z_1w_2u_1t_1\\
&+ (x_1+x_2)y_1z_1(w_1+w_2)u_2t_1 + (x_1+x_2)y_2z_2(w_1+w_2)u_1t_2
\end{aligned}$$

is an weakly minimal BNP of this type.

Minimal BNPs of small types. We proved in Proposition 4.11 that there are no minimal BNPs of type $(d, \ldots, d) \in \mathbb{N}^n$ when $2 \leq d \leq \frac{n+1}{2}$. This result is sharp when $n = 4$ and $n = 6$. We show a minimal BNP of type $(3, 3, 3, 3)$ in Example 4.5. Also, we have checked that the following polynomial is a minimal BNP of type $(4,4,4,4,4,4)$

$$\begin{aligned} &111111 + 112222 + 113333 + 114444 + 121234 + 122143 + 123412 + 124321 \\ &+131342 + 132431 + 133124 + 134213 + 141423 + 142314 + 143241 + 144132 \\ &+211432 + 212341 + 213214 + 214123 + 221313 + 222424 + 223131 + 224242 \\ &+231221 + 232112 + 233443 + 234334 + 241144 + 242233 + 243322 + 244411 \\ &+311243 + 312134 + 313421 + 314312 + 321122 + 322211 + 323344 + 324433 \\ &+331414 + 332323 + 333232 + 334141 + 341331 + 342442 + 343113 + 344224 \\ &+411324 + 412413 + 413142 + 414231 + 421441 + 422332 + 423223 + 424114 \\ &+431133 + 432244 + 433311 + 434422 + 441212 + 442121 + 443434 + 444343 \end{aligned}$$

($ijkpqr$ means $x_i y_j z_k w_p u_q t_r$). We do not know whether a minimal BNP of type $(4,4,4,4,4)$ exists or not.

Existence of BNSSs. The existence problem of BNSSs (Problem 1) looks much harder than that of BNPs (Problem 2). Indeed, we do not even know if a BNSSs of type (k,k,k,k) exists when $k \geq 4$. This paper unfortunately does not give much contributions to Problem 1, but we are hoping that ideas used in the paper are useful to study this problem. Indeed, the cross-polytopal ball in Example 4.16 was found by computing the reduced inverse system of the BNSS of type $(3,3,3,3)$ given in [14] and this example leads us a construction of BNSSs of types $(2, 4k-1, 4k-1, 4k-1)$ in section 6. We are expecting that a better understandings of BNPs helps studying the existence of BNSSs.

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