THE DEPTH OF AN IDEAL WITH A GIVEN HILBERT FUNCTION

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Abstract. Let $A = K[x_1, \ldots, x_n]$ denote the polynomial ring in n variables over a field K with each $\deg x_i = 1$. Let I be a homogeneous ideal of A with $I \neq A$ and $H_{A/I}$ the Hilbert function of the quotient algebra A/I. Given a numerical function $H : \mathbb{N} \to \mathbb{N}$ satisfying $H = H_{A/I}$ for some homogeneous ideal I of A, we write $\mathcal{A}_H$ for the set of those integers $0 \leq r \leq n$ such that there exists a homogeneous ideal I of A with $H_{A/I} = H$ and with $\operatorname{depth} A/I = r$. It will be proved that one has either $\mathcal{A}_H = \{0, 1, \ldots, b\}$ for some $0 \leq b \leq n$ or $|\mathcal{A}_H| = 1$.

Introduction

Let $A = K[x_1, \ldots, x_n]$ denote the polynomial ring in n variables over a field K with each $\deg x_i = 1$. Let I be a homogeneous ideal of A with $I \neq A$ and H_R the Hilbert function of the quotient algebra $R = A/I$. Thus $H_R(q)$, $q = 0, 1, 2, \ldots$, is the dimension of the subspace of R spanned over K by the homogeneous elements of R of degree q. A classical result [3, Theorem 4.2.10] due to Macaulay guarantees that, given a numerical function $H : \mathbb{N} \to \mathbb{N}$, where $\mathbb{N}$ is the set of nonnegative integers, there exists a homogeneous ideal I of A with $I \neq A$ such that H is the Hilbert function of the quotient algebra $R = A/I$ if and only if $H(0) = 1$, $H(1) \leq n$ and $H(q+1) \leq H(q)^{\langle q \rangle}$ for $q = 1, 2, \ldots$, where $H(q)^{\langle q \rangle}$ is defined in [3, p. 161].

Given a numerical function $H : \mathbb{N} \to \mathbb{N}$ satisfying $H(0) = 1$, $H(1) \leq n$ and $H(q+1) \leq H(q)^{\langle q \rangle}$ for $q = 1, 2, \ldots$, we write $\mathcal{A}_H$ for the set of those integers $0 \leq r \leq n$ such that there exists a homogeneous ideal I of A with $H_{A/I} = H$ and with $\operatorname{depth} A/I = r$. We will show that, given a numerical function $H : \mathbb{N} \to \mathbb{N}$ satisfying $H(0) = 1$, $H(1) \leq n$ and $H(q+1) \leq H(q)^{\langle q \rangle}$ for $q = 1, 2, \ldots$, one has (i) $\mathcal{A}_H = \{n - \delta\}$ if H is of the form (1) of Proposition 1.5 and (ii) $\mathcal{A}_H = \{0, 1, \ldots, b\}$ for some $b \geq 0$ if H cannot be of the form (1). The statement (i) will be proved in Theorems 1.6, and the statement (ii) will be proved in Theorem 2.1. Also, we will introduce a way to determine the integer $b = \max \mathcal{A}_H$ from H in Theorem 2.2.

1. Universal lexsegment ideals

Let $A = K[x_1, \ldots, x_n]$ denote the polynomial ring in n variables over a field K with each $\deg x_i = 1$ and $A_{[m]} = K[x_1, \ldots, x_{n+m}]$, where m is a positive integer. Work with the lexicographic order $<_{\text{lex}}$ on A induced by the ordering $x_1 > x_2 > \cdots > x_n$

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of the variables. Write, as usual, $G(I)$ for the (unique) minimal system of monomial generators of a monomial I of A. Recall that a monomial ideal I of A is *lexsegment* if, for a monomial u of A belonging to I and for a monomial v of A with $\deg u = \deg v$ and with $v >_{\text{lex}} u$, one has $v \in I$. A lexsegment ideal I of A is called *universal lexsegment* ([1]) if, for any integer $m \geq 1$, the monomial ideal $IA_{[m]}$ of the polynomial ring $A_{[m]}$ is lexsegment. In other words, a universal lexsegment ideal of A is a lexsegment ideal $I = (u_1, \ldots, u_t)$ of A which remains being lexsegment if we regard $I = (u_1, \ldots, u_t)$ as an ideal of the polynomial ring $A_{[m]}$ for all $m \geq 1$.

Example 1.1. (a) The lexsegment ideal $(x_1^2, x_1x_2^2)$ of $K[x_1, x_2]$ is universal lexsegment. In fact, the ideal $(x_1^2, x_1x_2^2)$ of $K[x_1, \ldots, x_m]$ is lexsegment for all $m \geq 2$.

(b) The lexsegment ideal $(x_1^3, x_1^2x_2, x_1x_2^2)$ of $K[x_1, x_2]$ cannot be universal lexsegment. Indeed, since $x_1x_2^2 <_{\text{lex}} x_1^2x_3$ in $K[x_1, x_2, x_3]$, the ideal $(x_1^3, x_1^2x_2, x_1x_2^2)$ of $K[x_1, x_2, x_3]$ is not lexsegment.

Proposition 1.2.

(a) *Let I be a lexsegment ideal of A with $G(I) = \{u_1, \ldots, u_\delta\}$ where $\deg u_1 \leq \cdots \leq \deg u_\delta$ and where $u_{i+1} <_{\text{lex}} u_i$ if $\deg u_i = \deg u_{i+1}$. Let $s_1 = \deg u_1 - 1$ and $s_i = \deg u_i - \deg u_{i-1}$ for $i = 2, 3, \ldots, \delta$. Then, for $k \leq n$, one has*

$$u_k = x_1^{s_1} x_2^{s_2} \cdots x_k^{s_k+1}.$$

(b) *Given an integer $1 \leq \delta \leq n$ together with a sequence of integers $1 \leq e_1 \leq \cdots \leq e_\delta$, there is a lexsegment ideal I of A with $G(I) = \{u_1, \ldots, u_\delta\}$ such that $\deg u_i = e_i$ for $i = 1, \ldots, \delta$.*

Proof. (a) Since $u_1 = x_1^{\deg u_1}$, one has $u_1 = x_1^{s_1+1}$. Let $1 < k \leq \min\{n, \delta\}$ and suppose that $u_{k-1} = x_1^{s_1} x_2^{s_2} \cdots x_{k-1}^{s_{k-1}+1}$. Since the ordering of $u_1, u_2, \ldots, u_\delta$ implies that the monomial ideal $(u_1, \ldots, u_{k-1})$ is lexsegment, the smallest monomial with respect to $<_{\text{lex}}$ of degree $\deg u_k$ belonging to $(u_1, \ldots, u_{k-1})$ is $u_{k-1}x_n^{s_k}$. Since u_k is the largest monomial with respect to $<_{\text{lex}}$ which satisfies $\deg u_k = \deg(u_{k-1}x_n^{s_k})$ and $u_k <_{\text{lex}} u_{k-1}x_n^{s_k}$, we have $u_k = (u_{k-1}/x_{k-1})x_k^{s_k+1}$. Thus $u_k = x_1^{s_1} x_2^{s_2} \cdots x_{k-1}^{s_{k-1}} x_k^{s_k+1}$, as desired.

(b) This can be easily done by induction on δ. Let $\delta \leq n$ and suppose that J is a lexsegment ideal of A with $G(J) = \{u_1, \ldots, u_{\delta-1}\}$ such that $\deg u_i = e_i$ for $i = 1, 2, \ldots, \delta - 1$. Then by (a) we have $G(J) \subset K[x_1, \ldots, x_{\delta-1}]$. Hence $x_\delta^{e_\delta} \notin J$. Thus there exists a monomial of degree e_δ which does not belong to J. Let u_δ be the largest monomial of degree e_δ with respect to $<_{\text{lex}}$ which does not belong to J. Then $(u_1, \ldots, u_{\delta-1}, u_\delta)$ is a lexsegment ideal of A. □

Corollary 1.3. *A lexsegment ideal I of A is universal lexsegment if and only if $|G(I)| \leq n$, where $|G(I)|$ is the number of monomials belonging to $G(I)$.*

Proof. Let $G(I) = \{u_1, \ldots, u_\delta\}$, where $\deg u_1 \leq \cdots \leq \deg u_\delta$. It $\delta \geq n + 1$, then $IA_{[1]}$ is not a lexsegment ideal of $A_{[1]}$ since Proposition 1.2 (a) says that, for any lexsegment ideal J of $A_{[1]}$ with $|G(J)| \geq n + 1$, there exists a generator $v \in G(J)$ such that x_{n+1} divides v. Thus I is not a universal lexsegment if $\delta \geq n + 1$.

Assume that $\delta \leq n$. For any positive integer m, Proposition 1.2 (b) says that there exists the lexsegment ideal J of $A_{[m]}$ such that $G(J) = \{v_1, \ldots, v_\delta\}$ satisfies

$\deg v_i = \deg u_i$ for $i = 1, 2, \ldots, \delta$. Then Proposition 1.2 (a) says that $G(I) = G(J)$. Thus $IA_{[m]}$ is a lexsegment ideal of $A_{[m]}$ for all $m \geq 1$ if $\delta \leq n$. □

For any monomial u of A, let $m(u)$ be the biggest integer $1 \leq i \leq n$ for which x_i divides u. A monomial ideal I of A is said to be *stable* if $u \in I$ implies $(x_q/x_{m(u)})u \in I$ for any $1 \leq q < m(u)$. Eliahou–Kervaire [5] says that, for a stable ideal I of A, the projective dimension $\operatorname{proj\,dim} A/I$ of the quotient algebra A/I coincides with $\max\{m(u) : u \in G(I)\}$. Since a lexsegment ideal is stable, it follows from Proposition 1.2 (a) together with the Auslander–Buchsbaum formula [3, Theorem 1.3.3] that

Corollary 1.4. *Let I be a lexsegment ideal of A and* $\operatorname{depth} A/I$ *the depth of the quotient algebra A/I of A. Then* $\operatorname{depth} A/I = \max\{n - |G(I)|, 0\}$.

It is known that, given a numerical function $H : \mathbb{N} \to \mathbb{N}$ satisfying $H(0) = 1$, $H(1) \leq n$ and $H(q+1) \leq H(q)^{\langle q \rangle}$ for $q = 1, 2, \ldots$, there exists the unique lexsegment ideal I of A with $H_{A/I} = H$. We say that a numerical function $H : \mathbb{N} \to \mathbb{N}$ satisfying $H(0) = 1$, $H(1) \leq n$ and $H(q + 1) \leq H(q)^{\langle q \rangle}$ for $q = 1, 2, \ldots$ is *critical* if the lexsegment ideal I of A with $H_{A/I} = H$ is universal lexsegment.

Proposition 1.5. *A numerical function $H : \mathbb{N} \to \mathbb{N}$ satisfying $H(0) = 1$, $H(1) \leq n$ and $H(q + 1) \leq H(q)^{\langle q \rangle}$ for $q = 1, 2, \ldots$ is critical if and only if there is an integer $1 \leq \delta \leq n$ together with a sequence of integers $(e_1, \ldots, e_\delta)$ with $1 \leq e_1 \leq \cdots \leq e_\delta$ such that*

$$H(q) = \binom{n-1+q}{n-1} - \sum_{i=1}^{\delta} \binom{n-i+q-e_i}{n-i} \tag{1}$$

for $q = 0, 1, \ldots$. Moreover, δ is equal to the number of minimal monomial generators of the universal lexsegment ideal I of A with $H_{A/I} = H$.

Proof. First, to prove the "only if" part, let I be a universal lexsegment ideal of A with $G(I) = \{u_1, \ldots, u_\delta\}$, where $\delta \leq n$. Suppose that $\deg u_1 \leq \cdots \leq \deg u_\delta$ and that $u_{i+1} <_{\text{lex}} u_i$ if $\deg u_i = \deg u_{i+1}$. Proposition 1.2 (a) says that, for $1 \leq i < j \leq \delta$, the monomial $x_i u_j$ is divided by u_i and no monomial belongs to both $u_i K[x_i, \ldots, x_n]$ and $u_j K[x_j, \ldots, x_n]$. Hence the direct sum decomposition $I = \bigoplus_{i=1}^{\delta} u_i K[x_i, \ldots, x_n]$ arises. Let $e_i = \deg u_i$ for $i = 1, 2, \ldots, \delta$. The fact that the number of monomials of degree q belonging to I is $\sum_{i=1}^{\delta} \binom{n-i+q-e_i}{n-i}$ yields the formula (1), as required.

Next we consider the "if" part. Let $H : \mathbb{N} \to \mathbb{N}$ be a numerical function of the form (1). Since $1 \leq e_1 \leq \cdots \leq e_\delta$ and $\delta \leq n$, Proposition 1.2 (b) and Corollary 1.3 say that there exists the universal lexsegment ideal I with $G(I) = \{u_1, \ldots, u_\delta\}$ such that $\deg(u_i) = e_i$ for all i. Then the computation of Hilbert functions in the proof of the "only if" part implies $H(I, q) = H(q)$ for all $q \in \mathbb{N}$. □

A *critical* ideal of A is a homogeneous ideal I of A with $I \neq A$ such that the Hilbert function H_R of the quotient algebra $R = A/I$ is critical. In other words, a critical ideal of A is a homogeneous ideal I of A such that the lexsegment ideal I^{lex} is universal lexsegment, where I^{lex} is the unique lexsegment ideal of A such that A/I and A/I^{lex} have the same Hilbert function. Somewhat surprisingly,

Theorem 1.6. *Suppose that a homogeneous ideal I of A is critical. Then*

$$\operatorname{depth} A/I = \operatorname{depth} A/I^{\text{lex}}.$$

Proof. Let β_{ij} (resp. β'_{ij}) denote the graded Betti numbers of I (resp. I^{lex}). Let $G(I^{\text{lex}}) = \{u_1, \ldots, u_\delta\}$ with $\delta \leq n$, where $\deg u_1 \leq \cdots \leq \deg u_\delta$ and where $u_{i+1} <_{\text{lex}} u_i$ if $\deg u_i = \deg u_{i+1}$. Let $e_i = \deg u_i$ for $i = 1, \ldots, \delta$. Eliahou–Kervaire [5] together with Proposition 1.2 (a) guarantees that $\beta'_{i,\delta-1+e_\delta} = 0$ unless $i = \delta - 1$ and $\beta'_{\delta-1,\delta-1+e_\delta} = 1$. Since A/I and A/I^{lex} have the same Hilbert function, it follows from [3, Lemma 4.1.13] that

$$\sum_{i\geq 0}(-1)^i\beta_{i,\delta-1+e_\delta} = \sum_{i\geq 0}(-1)^i\beta'_{i,\delta-1+e_\delta}.$$

Since $\beta_{ij} \leq \beta'_{ij}$ for all i and j ([2], [7] and [8]), it follows that $\beta_{\delta-1,\delta-1+e_\delta} = 1$. Thus in particular $\operatorname{proj\,dim} A/I \geq \delta$. Since $\operatorname{proj\,dim} A/I^{\text{lex}} = \delta$ and $\operatorname{proj\,dim} A/I \leq \operatorname{proj\,dim} A/I^{\text{lex}}$, it follows that $\operatorname{proj\,dim} A/I = \operatorname{proj\,dim} A/I^{\text{lex}} = \delta$. Thus we have $\operatorname{depth} A/I = \dim A/I^{\text{lex}} = n - \delta$, as desired. ☐

Moreover, in case of monomial ideals, the graded Betti numbers of a critical ideal are determined by its Hilbert function.

Corollary 1.7. *Suppose that a monomial ideal I of A is critical. Then I and I^{lex} have the same graded Betti numbers.*

Proof. It follows from Taylor's resolution of monomial ideals (see [5, p. 18]) that

$$\text{proj dim}(A/I) \leq |G(I)|.$$

On the other hand, Corollary 1.4 and Theorem 1.6 say that

$$\text{proj dim}(A/I) = \text{proj dim}(A/I^{\text{lex}}) = |G(I^{\text{lex}})|.$$

Since the number of elements in $G(I^{\text{lex}})$ is always larger than that of $G(I)$, we have $|G(I)| = |G(I^{\text{lex}})|$. This means $\sum_{j\geq 0}\beta_{0j}(I) = \sum_{j\geq 0}\beta_{0j}(I^{\text{lex}})$. Then it follows from [4, Theorem 1.3] that $\beta_{ij}(I) = \beta_{ij}(I^{\text{lex}})$ for all i and j. ☐

We are not sure that Corollary 1.7 holds for an arbitrary critical ideal.

Example 1.8. Let I be the monomial ideal (x_1x_4, x_3x_4) of $K[x_1, x_2, x_3, x_4]$. Since $I^{\text{lex}} = (x_1^2, x_1x_2)$ is universal lexsegment, it follows that $\operatorname{depth} A/I = 2$.

2. Depth and Hilbert functions

Let, as before, $A = K[x_1, \ldots, x_n]$ denote the polynomial ring in n variables over a field K with each $\deg x_i = 1$. Given a numerical function $H : \mathbb{N} \to \mathbb{N}$ satisfying $H(0) = 1$, $H(1) \leq n$ and $H(q+1) \leq H(q)^{\langle q\rangle}$ for $q = 1, 2, \ldots$, we write $\mathcal{A}_H$ for the set of those integers $0 \leq r \leq n$ such that there exists a homogeneous ideal I of A with $H_{A/I} = H$ and with $\operatorname{depth} A/I = r$. It follows from Corollary 1.4 together with Theorem 1.6 that if H is critical, that is, H is of the form (1), then $\mathcal{A}_H = \{n - \delta\}$.

Theorem 2.1. *Suppose that a numerical function $H : \mathbb{N} \to \mathbb{N}$ satisfying $H(0) = 1$, $H(1) \leq n$ and $H(q+1) \leq H(q)^{\langle q \rangle}$ for $q = 1, 2, \ldots$ is noncritical. Then $\mathcal{A}_H = \{0, 1, 2, \ldots, b\}$, where b is the biggest integer for which $b \in \mathcal{A}_H$.*

Proof. We may assume that K is infinite. Let I be a homogeneous ideal of A with $H_{A/I} = H$ and with $\operatorname{depth} A/I = b$. Let $0 \leq r \leq b$. Since K is infinite and since $\operatorname{depth} A/I = b$, there exists a regular sequence $(\theta_1, \ldots, \theta_r)$ of A/I with each $\deg \theta_i = 1$. It then follows that there exists a homogeneous ideal J of $B = K[x_1, \ldots, x_{n-r}]$ such that the ideal JA of A satisfies $H_{A/(JA)} = H$.

We now claim that the lexsegment ideal $J^{\text{lex}} \subset B$ of J cannot be universal lexsegment. In fact, if J^{lex} is universal lexsegment, then J^{lex} remains being lexsegment in the polynomial ring $K[x_1, \ldots, x_m]$ for each $m \geq n - r$. In particular the ideal $J^{\text{lex}}A$ of A is universal lexsegment. Since $H_{A/(JA)} = H_{A/(J^{\text{lex}}A)} = H$, the numerical function H is critical, a contradiction.

Since the lexsegment ideal J^{lex} of J cannot be universal lexsegment, it follows from Corollaries 1.3 and 1.4 that $\operatorname{depth} B/J^{\text{lex}} = 0$. Thus $\operatorname{depth} A/(J^{\text{lex}}A) = r$. Hence $r \in \mathcal{A}_H$, as desired. □

One may ask a way to compute the integer $b = \max \mathcal{A}_H$ from H. This integer b can be determined as follows: Let $H : \mathbb{N} \to \mathbb{N}$ be a numerical function. The *differential* $\Delta^1(H)$ of H is the numerical function defined by $\Delta^1(H)(0) = 1$ and $\Delta^1(q) = H(q) - H(q-1)$ for $q \geq 1$. We define *p-th differential* $\Delta^p(H) = \Delta^1(\Delta^{p-1}(H))$ inductively.

Theorem 2.2. *Let $H : \mathbb{N} \to \mathbb{N}$ be a numerical function satisfying $H(0) = 1$, $H(1) \leq n$ and $H(q+1) \leq H(q)^{\langle q \rangle}$ for all $q \geq 1$. Then one has*

$$(2)\quad \max A_H = \max\{p : \Delta^p(H) \text{ satisfies } \Delta^p(H)(q+1) \leq \Delta^p(H)(q)^{\langle q \rangle} \text{ for } q \geq 1\}$$

Proof. If p is an integer which belongs to the righthand side of (2), then there exists a homogeneous ideal J of $B = K[x_1, \ldots, x_{n-p}]$ such that $H_{B/J} = \Delta^p(H)$. Recall that if M is a graded R-module and $\vartheta_1, \ldots, \vartheta_r$ with each $\deg(\vartheta_i) = 1$ is a regular sequence of M, then $H_{M/(\vartheta_1, \ldots, \vartheta_r)M} = \Delta^p(H_M)$. Set $M = A/(JA)$. Then, since $x_n, x_{n-1}, \ldots, x_{n-p+1}$ is a regular sequence of $A/(JA)$ and $M/(x_n, \ldots, x_{n-p+1})M \cong B/J$, we have $H_{A/(JA)} = H$ and $\operatorname{depth}(A/(JA)) \geq p$. This says that the lefthand side of (2) is larger than or equal to the righthand side of (2).

On the other hand, if there exists a homogeneous ideal I of A such that $H = H_{A/I}$ and $\operatorname{depth}(A/I) = p$, then, in the same way as Theorem 2.1, there exists a homogeneous ideal J of $B = K[x_1, \ldots, x_{n-p}]$ such that $H_{A/(JA)} = H$ and $H_{B/J} = \Delta^p(H)$. Thus the lefthand side of (2) is smaller than or equal to the righthand side of (2). □

Example 2.3. Let I be the monomial ideal $(x_1x_4, x_1x_5, x_2x_5, x_3x_5, x_2x_3x_4)$ of $A = K[x_1, x_2, x_3, x_4, x_5]$. Then

$$I^{\text{lex}} = (x_1^2, x_1x_2, x_1x_3, x_1x_4, x_1x_5^2, x_2^3, x_2^2x_3, x_2^2x_4^2, x_2^2x_4x_5, x_2^2x_5^3, x_2x_3^4, x_2x_3^3x_4^2).$$

Thus $\operatorname{depth} A/I^{\text{lex}} = 0$ by Corollary 1.4. Since the Hilbert series $\sum_{q=0}^{\infty} H_{A/I}(q)\lambda^q$ of A/I is $(1 + 2\lambda - \lambda^2 - \lambda^3)/(1 - \lambda)^3$, it follows from [3, Corollary 4.1.10] that the Krull dimension of A/I is 3 and $3 \notin \mathcal{A}_H$. Since $\operatorname{depth} A/I = 2$, one has $\mathcal{A}_H = \{0, 1, 2\}$.

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