Perfectly contractile graphs and quadratic toric rings

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Perfect graph

G : a finite simple graph
(no loops and no multiple edges)
with vertex set $[d] = \{1, 2, \ldots, d\}$ and edge set E

A clique in G is a set of pairwise adjacent vertices in G.

$\omega(G) :=$ the clique number of G
$= \max\{|C| : C$ is a clique of $G\}$

$\chi(G) :=$ the chromatic number of G

In general,

$$\omega(G) \leq \chi(G)$$

Definition

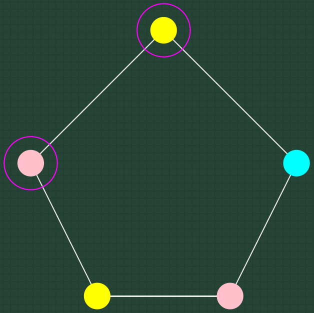
We say that G is perfect if for any induced subgraph H of G,

$$\omega(H) = \chi(H)$$

e.g., bipartite graph, chordal graph

Example I

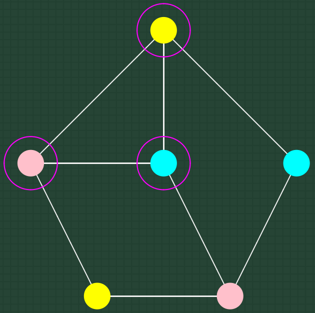
G:



$$\omega(G) = 2 < \chi(G) = 3$$

Example II

G:



$$\omega(G) = \chi(G) = 3$$

But G is NOT perfect.

Perfect Graph Theorem

$\overline{G}$:= the complement graph of G

Theorem (Weak Perfect Graph Theorem, Lovàsz)

G is perfect if and only if $\overline{G}$ is perfect.

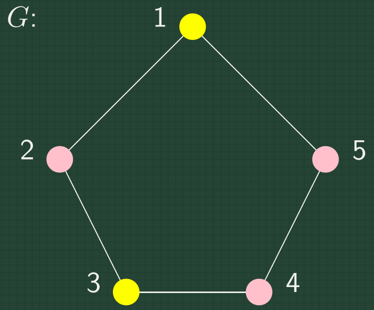
An odd hole is an induced odd cycle of length ≥ 5.
An odd antihole is the complement graph of an odd hole.

Theorem (Strong Perfect Graph Theorem, Chudnovsky–Robertson–Seymour–Thomas)

G is perfect if and only if G contains no odd holes and no odd antiholes as induced subgraphs.

Stable set

$S \subset [d]$ is a stable set or an independent set of G if for $\forall i, j \in S$, $\{i, j\} \notin E$.
$S(G) :=$ the set of stable sets of G.



$\{1, 3\}$ is stable. $\{2, 4, 5\}$ is NOT stable.

$$S(G) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1,3\}, \{1,4\}, \{2,4\}, \{2,5\}\}$$

Algebraic Characterization of Perfect Graph I

K : field.
$K[\mathbf{t}^{\pm 1}, s] := K[t_1^{\pm 1}, \ldots, t_d^{\pm 1}, s]$.
$K[G] := K[(\prod_{i \in S} t_i)s : S \in S(G)] \subset K[\mathbf{t}^{\pm 1}, s]$.
$R[G] := K[x_S : S \in S(G)]$ with deg $x_S = 1$.
$\pi : R(G) \to K[G]$ defined by $x_S \mapsto (\prod_{i \in S} t_i)s$.
$I_G = \ker \pi$.

Theorem (Ohsugi–Hibi)

TFAE:

1. *G is perfect;*
2. *The initial ideal of I_G with respect to any reverse lexicographic order is squarefree;*
3. *The initial ideal of I_G with respect to a reverse lexicographic order such that $x_\emptyset$ is the smallest variable is squarefree.*

Algebraic Characterization of Perfect Graph II

$K[\Gamma(G)] := K[(\prod_{i\in S} t_i)s, (\prod_{i\in S} t_i^{-1})s : S \in S(G)]$.
$K[\Omega(G)] := K[(\prod_{i\in S} t_i)us, (\prod_{i\in S} t_i^{-1})u^{-1}s, s : S \in S(G)]$.

Theorem (Ohsugi–Hibi, Hibi–T)

TFAE:

1. G *is perfect;*
2. $\overline{K[\Gamma(G)]}$ *is (normal) Gorenstein;*
3. $K[\Gamma(G)]$ *is normal Gorenstein;*
4. $K[\Omega(G)]$ *is normal;*
5. $K[\Omega(G)]$ *is normal Gorenstein.*

Quadratic toric rings

G : a perfect graph.

Question

When is I_G generated by quadratic binomials? When does I_G possess a quadratic initial ideal?

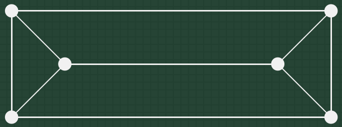
e.g.,

- comparability graphs;
- almost bipartite graphs;
- chordal graphs;
- ring graphs;
- the complement graphs of chordal bipartite graphs.

Even antihole

An even hole is an induced even cycle of length ≥ 6.
An even antihole is the complement graph of an even hole.

$\overline{C_6}$:



Proposition

Let G be a perfect graph. If I_G is generated by quadratic binomials, then G contains no even antiholes.

Odd stretcher

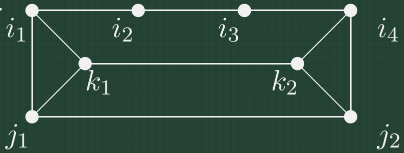
An odd stretcher $G_{s,t,u}$ is a graph on the vertex set

$$\{i_1, i_2, \ldots, i_{2s}, j_1, j_2, \ldots, j_{2t}, k_1, k_2, \ldots, k_{2u}\}$$

with edges

$$\begin{gathered}\{i_1, j_1\}, \{i_1, k_1\}, \{j_1, k_1\}, \{i_{2s}, j_{2t}\}, \{i_{2s}, k_{2t}\}, \{j_{2t}, k_{2s}\}, \\ \{i_1, i_2\}, \{i_2, i_3\}, \ldots, \{i_{2s-1}, i_{2s}\}, \\ \{j_1, j_2\}, \{j_2, j_3\}, \ldots, \{j_{2t-1}, j_{2t}\}, \\ \{k_1, k_2\}, \{k_2, k_3\}, \ldots, \{k_{2u-1}, k_{2u}\}.\end{gathered}$$

$G_{2,1,1}$:



Proposition

Let G be a perfect graph. If I_G is generated by quadratic binomials, then G contains no odd stretchers as induced subgraphs.

Perfectly contractile graph

An even pair in a graph G is a pair of non-adjacent vertices of G such that the length of all chordless paths between them is even.

Contracting a pair of vertices $\{x, y\}$ an in a graph G means removing x and y and adding a new vertex z with edges to every neighbor of x or y.

A graph G is called even contractile if there is a sequence $G_0, \ldots, G_k$ of graphs such that $G = G_0$, each G_i is obtained from G_{i-1} by contracting an even pair of G_{i-1}, and G_k is a complete graph.

Definition (Bertschi)

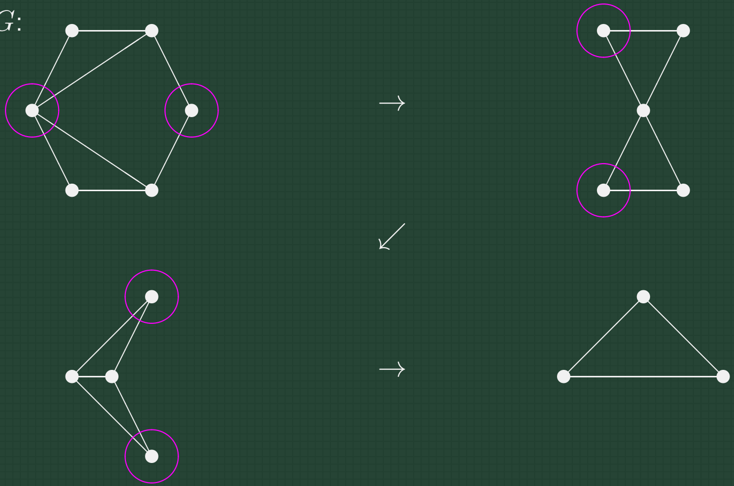
We say that G is perfectly contractile if any induced subgraphs of G are even contractile.

Theorem (Bertschi)

Every perfectly contractile graph is perfect.

Example I

G:

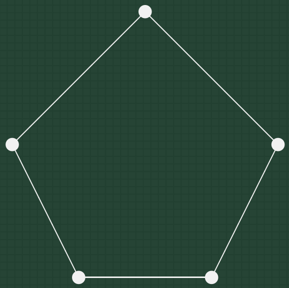


G is even contractile.

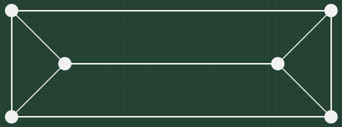
In fact, G is perfectly contractile.

Example II

C_5:



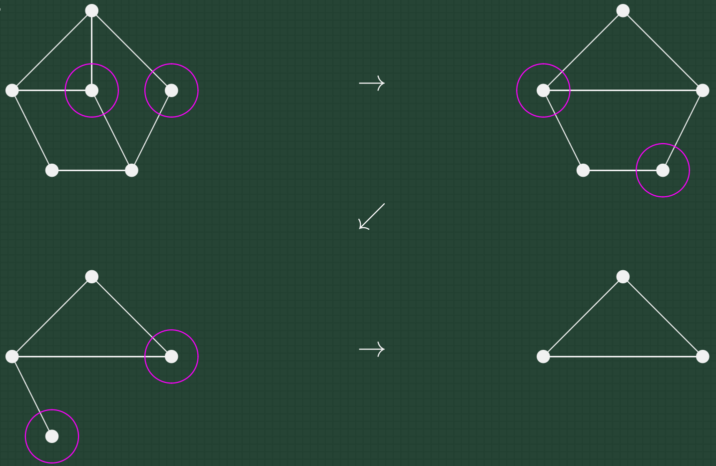
$\overline{C_6}$:



C_5 and $\overline{C_6}$ have no even pairs, hence, they are NOT even contractile.

Example III

G:



G is even contractile.

But G is NOT perfectly contractile.

Combinatorial characterization of perfectly contractile graph (conjecture)

Conjecture (Everett–Reed)

G is perfectly contractile if and only if G contains no odd holes, no antiholes and no odd stretchers as induced subgraphs.

Proposition

If G is perfectly contractile, then G contains no odd holes, no antiholes and no odd stretchers as induced subgraphs.

Algebraic characterization of perfectly contractile graph (conjecture)

Proposition

Let G be a perfect graph. If I_G is generated by quadratic binomials, then G contains no even antiholes and no odd stretchers as induced subgraphs.

Conjecture

Let G be a perfect graph. TFAE:

1. *G is perfectly contractile;*
2. *I_G is generated by quadratic binomials;*
3. *G contains no even antiholes and no odd stretchers as induced subgraphs.*

Meyniel graph

Definition

A graph is called Meyniel or very strongly perfect if any odd cycle of length ≥ 5 has at least two chords.

Theorem (Bertschi)

Every Meyniel graph is perfectly contractile.

Theorem (Ohsugi–Shibata–T)

For each Meyniel graph G, I_G is generated by quadratic binomials.

Perfectly orderable graph

G : a graph on the vertex set $\{v_1, \ldots, v_n\}$.
An ordering $v_1 < \cdots < v_n$ of the vertex set
of G is called perfect if G contains no P_4 $abcd$ such that $a < b$ and $d < c$.

P_4 $abcd$:

Definition

We say that G is perfect orderable if it has a perfect ordering
$v_1 < \cdots < v_n$ of the vertex set.

Theorem (Bertschi)

Every perfectly orderable graph is perfectly contractile.

Perfectly orderable graph

Theorem (Ohsugi–Shibata–T)

For any perfectly orderable graph G, the initial ideal of I_G with respect to a reverse lexicographic order is squarefree and quadratic.

Remark

The following graphs are perfectly orderable:

- *comparability graphs;*
- *chordal graphs;*
- *the complement graphs of chordal graphs.*

Hence this theorem is a generalization of results on several toric ideals.

Clique separable graph

Definition

We say that a graph is clique separable if it is obtained by successive gluing along cliques starting with graphs of Type 1 or 2:

1. The join of a bipartite graph with more than 3 vertices with a complete graph;
2. A complete multipartite graph.

Theorem (Bertschi)

Every clique separable graph is perfectly contractile.

Theorem (Ohsugi–Shibata–T)

For any clique separable graph G, the initial ideal of I_G with respect to a reverse lexicographic order is squarefree and quadratic.