分割 $(n-2,2)$ と $(d,d,1)$ に対する Specht ideal の極小自由分解

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Let $n \in \mathbb{Z}_{>0}$, and set $[n] := \{1, \cdots, n\}$.
Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ be a partition of n, with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l \geq 1$ and $\sum_{i=1}^{l} \lambda_i = n$.
The *(Young) tableau* of shape λ is a bijection from $[n]$ to the set of boxes in the Young diagram of λ.
$\mathrm{Tab}(\lambda)$: the set of all Young tableaux of shape λ.

Example 1.1

The following is a tableau of shape $(4, 2, 1)$.

3	5	1	7
6	2		
4			

Definition 1.2

$T \in \mathrm{Tab}(\lambda)$ is a *standard tableau of* λ*, if all columns (resp. rows) are increasing from top to bottom (resp. from left to right).*
The set of all standard tableaux of λ *is denoted by* $\mathrm{SYT}(\lambda)$*.*

Example 1.3

$$T = \begin{array}{|c|c|c|c|}\hline 3 & 5 & 1 & 7 \\ \hline 6 & 2 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}, \qquad T' = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 5 & 7 \\ \hline 3 & 4 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}.$$

Then T is not standard, and T' is standard.

Definition 1.4

Let $T_1, T_2 \in \mathrm{Tab}(\lambda)$. T_1 and T_2 are *row equivalent*, $T_1 \sim T_2$, if corresponding rows of two tableaux contain the same elements.
A tabloid of λ is

$$\{T\} := \{T' \mid T \sim T'\}$$

where $T \in \mathrm{Tab}(\lambda)$.

Example 1.5

$$T = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 4 & 2 \\ \hline \end{array},$$

then

$$\{T\} = \left\{ \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 4 & 2 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 4 & 2 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 & 4 \\ \hline \end{array} \right\}$$

Definition 1.6

Let $T \in \mathrm{Tab}(\lambda)$ has columns $C_1, \ldots, C_k$, and set $S(C_i)$ be the set of permutations of elements of C_i. Then

$$C(T) := S(C_1) \times \cdots \times S(C_k).$$

For $T \in \mathrm{Tab}(\lambda)$, set

$$e(T) := \sum_{\sigma \in C(T)} \mathrm{sgn}(\sigma)\sigma\{T\}.$$

Example 1.7

$$e\left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 4 & 2 \\ \hline\end{array}\right) = \left\{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 4 & 2 \\ \hline\end{array}\right\} - \left\{\begin{array}{|c|c|}\hline 4 & 3 \\ \hline 1 & 2 \\ \hline\end{array}\right\} - \left\{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & 3 \\ \hline\end{array}\right\} + \left\{\begin{array}{|c|c|}\hline 4 & 2 \\ \hline 1 & 3 \\ \hline\end{array}\right\}$$

Definition 1.8

Define the K-vector space as follows.

$$V_\lambda := K\langle e(T) \mid T \in \mathrm{Tab}(\lambda)\rangle$$

This is called the Specht module of λ.

Remark 1.9

V_λ is an $\mathfrak{S}_n$-module, and we have

$$V_\lambda = K\langle e(T) \mid T \in \mathrm{SYT}(\lambda)\rangle$$

Classical fact 1.10

If $\mathrm{char}(K) = 0$*, the Specht modules* V_λ *for partitions* λ *of* n *are irreducible, and form a complete list of irreducible representations of* $\mathfrak{S}_n$*.*

$R := K[x_1, \cdots, x_n]$: polynomial ring over a field K.
$T \in \mathrm{Tab}(\lambda)$.

If the j-th column of T consists of $j_1, j_2, \ldots, j_m$ in the order from top to bottom, then

$$f_T(j) := \prod_{1 \leq s < t \leq m} (x_{j_s} - x_{j_t}) \in R$$

(if the j-th column has only one box, then we set $f_T(j) = 1$).
The *Specht polynomial* f_T of T is given by

$$f_T := \prod_{j=1}^{\lambda_1} f_T(j).$$

Example 1.11

If T is the tableau

$$\begin{array}{|c|c|c|c|}\hline 3 & 5 & 1 & 7 \\ \hline 6 & 2 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array},$$

then $f_T = (x_3 - x_6)(x_3 - x_4)(x_6 - x_4)(x_5 - x_2)$.

Classical fact 1.12

As $\mathfrak{S}_n$-modules,

$$\begin{array}{ccc} V_\lambda & \cong & K\langle f_T \mid T \in \mathrm{Tab}(\lambda)\rangle \\ e(T) & \longmapsto & f_T \end{array}$$

Definition 1.13

The *ideal*

$$I_\lambda^{\mathrm{Sp}} := (\, f_T \mid T \in \mathsf{Tab}(\lambda)) \subset R$$

is called the Specht ideal of λ.

Remark 1.14

$$I_\lambda^{\mathrm{Sp}} = (f_T \mid T \in \mathsf{SYT}(\lambda))$$

Theorem 1.15 (J.Watanabe-Yanagawa, 2019, Y, 2019)

If $R/I_\lambda^{\mathrm{Sp}}$ is Cohen–Macaulay(CM for short), then one of the following conditions holds.

(1) $\lambda = (n-d, 1, \ldots, 1)$ $(= (n-d, 1^d))$,

(2) $\lambda = (n-d, d)$,

(3) $\lambda = (d, d, 1)$.

If $\mathrm{char}(K) = 0$, *the converse is also true.*

Example 1.16

For $\lambda = (n-3, 3)$, then

$$R/I^{\mathrm{Sp}}_{(n-3,3)} \text{ is CM} \iff \operatorname{char}(K) \neq 2.$$

Theorem 1.17 (Y,2019)

$R/I^{\mathrm{Sp}}_{(n-2,2)}$ is Gorenstein over any K.

Theorem 1.18 (S-Y,2020)

If $\operatorname{char}(K) = 0$, then $I^{\mathrm{Sp}}_{(d,d,1)}$ has $(d+2)$-linear resolution.

Inspired by comments of Prof. Murai, we will construct the minimal free resolutions in these cases.

For $R/I^{\mathrm{Sp}}_{(n-2,2)}$, we define the chain complex

$$\mathcal{F}_\bullet^{(n-2,2)} : 0 \longrightarrow F_{n-2} \xrightarrow{\partial_{n-2}} F_{n-3} \xrightarrow{\partial_{n-3}} \cdots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \longrightarrow 0 \qquad (2.1)$$

of graded free R-modules as follows. Here $F_0 = R$,
$F_1 = V_{(n-2,2)} \otimes_K R(-2)$,

$$F_i = V_{(n-1-i,2,1^{i-1})} \otimes_K R(-1-i)$$

for $1 \leq i \leq n-3$, and $F_{n-2} = V_{(1^n)} \otimes_K R(-n)$. For $T \in \mathrm{Tab}(n-2,2)$, set $\partial_1(e(T) \otimes 1) := f_T \in R = F_0$. To describe ∂_i for $2 \leq i \leq n-3$, we need preparation.

$$T = \begin{array}{|c|c|c|c|c|c|}
\hline
a_1 & b_1 & c_1 & c_2 & \cdots & c_{n-3-i} \\
\hline
a_2 & b_2 & \multicolumn{4}{c}{} \\
\cline{1-2}
\vdots & \multicolumn{5}{c}{} \\
\cline{1-1}
a_{i+1} & \multicolumn{5}{c}{} \\
\cline{1-1}
\end{array} \in \mathrm{Tab}(n-1-i, 2, 1^{i-1})$$

For j with $1 \leq j \leq i+1$, set

$$
T_j := \begin{array}{|c|c|c|c|c|c|c|}
\hline
a_1 & b_1 & c_1 & c_2 & \cdots & c_{n-3-i} & a_j \\
\hline
a_2 & b_2 \\
\cline{1-2}
\vdots \\
\cline{1-1}
a_{j-1} \\
\cline{1-1}
a_{j+1} \\
\cline{1-1}
\vdots \\
\cline{1-1}
a_{i+1} \\
\cline{1-1}
\end{array} \in \mathrm{Tab}(n-i, 2, 1^{i-2}).
$$

Then we have

$$
\partial_i(e(T) \otimes 1) := \sum_{j=1}^{i+1} (-1)^{j-1} e(T_j) \otimes x_{a_j} \in V_{(n-i,2,1^{i-2})} \otimes_K R(-i) = F_{i-1}.
$$

Similarly, for

$$T = \begin{array}{|c|}\hline a_1 \\ \hline a_2 \\ \hline \vdots \\ \hline a_n \\ \hline \end{array} \in \mathrm{Tab}(1^n)$$

and j, k with $1 \leq j < k \leq n$, set

$$T_{j,k} = \begin{array}{|c|c|}\hline \vdots & a_j \\ \hline \vdots & a_k \\ \hline \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \in \mathrm{Tab}(2, 2, 1^{n-4}),$$

where the first column is the "transpose" of

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline a_1 & a_2 & \cdots & a_{j-1} & a_{j+1} & \cdots & a_{k-1} & a_{k+1} & \cdots & a_n \\ \hline \end{array}.$$

Then

$$\begin{aligned} \partial_{n-2}(e(T) \otimes 1) &:= \sum_{1 \leq j < k \leq n} (-1)^{j+k-1} e(T_{j,k}) \otimes x_{a_j} x_{a_k} \\ &\in V_{(2,2,1^{n-4})} \otimes_K R(-n+2) = F_{n-3}. \end{aligned}$$

Theorem 2.1

If $\mathrm{char}(K) = 0$*, the complex* $\mathcal{F}_\bullet^{(n-2,2)}$ *of* (2.1) *is a minimal free resolution of* $R/I^{\mathrm{Sp}}_{(n-2,2)}$.

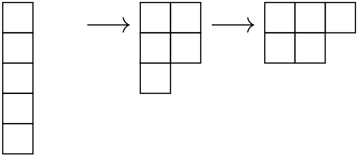
Theorem 2.1

If $\mathrm{char}(K) = 0$, *the complex* $\mathcal{F}_\bullet^{(n-2,2)}$ *of* (2.1) *is a minimal free resolution of* $R/I^{\mathrm{Sp}}_{(n-2,2)}$.

Example 2.2

We introduce $\mathcal{F}_\bullet^{(3,2)}$.

$$0 \longrightarrow R(-5) \longrightarrow R(-3)^5 \longrightarrow R(-2)^5 \longrightarrow R \longrightarrow R/I^{\mathrm{Sp}}_{(3,2)} \longrightarrow 0$$



$$\partial_3\left(e\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline 4\\\hline 5\\\hline\end{array}\right)\otimes 1\right) = e\left(\begin{array}{|c|c|}\hline 3&1\\\hline 4&2\\\hline 5\\\cline{1-1}\end{array}\right)\otimes x_1x_2 - e\left(\begin{array}{|c|c|}\hline 2&1\\\hline 4&3\\\hline 5\\\cline{1-1}\end{array}\right)\otimes x_1x_3 + e\left(\begin{array}{|c|c|}\hline 2&1\\\hline 3&4\\\hline 5\\\cline{1-1}\end{array}\right)\otimes x_1x_4$$

$$-e\left(\begin{array}{|c|c|}\hline 2&1\\\hline 3&5\\\hline 4\\\cline{1-1}\end{array}\right)\otimes x_1x_5 + e\left(\begin{array}{|c|c|}\hline 1&2\\\hline 4&3\\\hline 5\\\cline{1-1}\end{array}\right)\otimes x_2x_3 - e\left(\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline 5\\\cline{1-1}\end{array}\right)\otimes x_2x_4$$

$$+e\left(\begin{array}{|c|c|}\hline 1&2\\\hline 3&5\\\hline 4\\\cline{1-1}\end{array}\right)\otimes x_2x_5 + e\left(\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline 5\\\cline{1-1}\end{array}\right)\otimes x_3x_4 - e\left(\begin{array}{|c|c|}\hline 1&3\\\hline 2&5\\\hline 4\\\cline{1-1}\end{array}\right)\otimes x_3x_5$$

$$+e\left(\begin{array}{|c|c|}\hline 1&4\\\hline 2&5\\\hline 3\\\cline{1-1}\end{array}\right)\otimes x_4x_5$$

$$\partial_2\left(e\left(\begin{array}{|c|c|}\hline 3&1\\\hline 4&2\\\hline 5\\\cline{1-1}\end{array}\right)\otimes 1\right) = e\left(\begin{array}{|c|c|c|}\hline 4&1&3\\\hline 5&2\\\cline{1-2}\end{array}\right)\otimes x_3 - e\left(\begin{array}{|c|c|c|}\hline 3&1&4\\\hline 5&2\\\cline{1-2}\end{array}\right)\otimes x_4 + e\left(\begin{array}{|c|c|c|}\hline 3&1&5\\\hline 4&2\\\cline{1-2}\end{array}\right)\otimes x_5$$

Outline of proof

- $\partial_{i-1}\partial_i = 0$
- $\beta_i(R/I^{\mathrm{Sp}}_{(n-2,2)}) = \dim_K V_{(n-2-(i-1),2,1^{i-1})}$ for $i \geq 2$.
- Regard F_i as an $\mathfrak{S}_n$-module as follows. For $v \otimes f \in F_i = V_\lambda \otimes R(-j)$ and $\sigma \in \mathfrak{S}_n$, set $\sigma(v \otimes f) := \sigma v \otimes \sigma f \in F_i$.
  Then $\partial_i : F_i \longrightarrow F_{i-1}$ is an $\mathfrak{S}_n$-homomorphism. Where λ is a suitable partition of n, and j is a suitable integer.

- So is its restriction

$$[\partial_i]_j : [F_i]_j = V_\lambda \otimes_K [R(-j)]_j = V_\lambda \longrightarrow V_{\lambda'} \otimes_K R_l = [F_{i-1}]_j,$$

  where $l = 1$ if $2 \leq i \leq n-3$, and $l = 2$ if $i = 1, n-2$.
  Since $V_\lambda \otimes_K K \cong V_\lambda$ is irreducible as an $\mathfrak{S}_n$-module and $[\partial_i]_j$ is nonzero, we have $[\partial_i]_j$ is injective.

- $\mu(\operatorname{Ker} \partial_{i-1}) = \beta_{i,j}(R/I^{\mathrm{Sp}}_{(n-2,2)}) = \dim_K V_\lambda = \dim_K [\operatorname{Im} \partial_i]_j$ for $i \geq 2$.
  So $\mathcal{F}_\bullet^{(n-2,2)}$ is exact.

For $R/I^{\mathrm{Sp}}_{(d,d,1)}$, we define the chain complex

$$\mathcal{F}^{(d,d,1)}_{\bullet} : 0 \longrightarrow F_d \xrightarrow{\partial_d} F_{d-1} \xrightarrow{\partial_{d-1}} \cdots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \longrightarrow 0$$

of graded free R-modules as follows. Here $F_0 = R$ and

$$F_i = V_{(d,d-i+1,1^i)} \otimes_K R(-d-i-1)$$

for $1 \le i \le d$. As before, set $\partial_1(e(T) \otimes 1) := f_T \in R = F_0$. To describe ∂_i for $i \ge 2$, we need preparation. For

$$T = \begin{array}{|c|c|c|c|c|c|c|} \hline a_1 & b_2 & \cdots & b_{d-i+1} & b_{d-i+2} & \cdots & b_d \\ \hline a_2 & c_2 & \cdots & c_{d-i+1} \\ \cline{1-4} \vdots \\ \cline{1-1} a_{i+2} \\ \cline{1-1} \end{array} \in \mathrm{Tab}(d, d-i+1, 1^i)$$

For j with $1 \leq j \leq i+2$, set

$$T_j := \begin{array}{|c|c|c|c|c|c|c|c|}
\hline
a_1 & b_2 & \cdots & b_{d-i+1} & b_{d-i+2} & b_{d-i+3} & \cdots & b_d \\
\hline
a_2 & c_2 & \cdots & c_{d-i+1} & a_j \\
\cline{1-5}
\vdots \\
\cline{1-1}
a_{j-1} \\
\cline{1-1}
a_{j+1} \\
\cline{1-1}
\vdots \\
\cline{1-1}
a_{i+2} \\
\cline{1-1}
\end{array} \in \mathrm{Tab}(d, d-i+2, 1^{i-1})$$

Then we have

$$\begin{aligned}\partial_i(e(T)\otimes 1) &= \sum_{j=1}^{i+2}\sum_{\sigma\in H}(-1)^{j-1}e(\sigma(T_j))\otimes x_{a_j} \\ &\in V_{(d,d-i+2,1^{i-1})}\otimes_K R(-d-i) = F_{i-1}\end{aligned}$$

for $i \geq 3$, where H is the set of permutations of $\{b_{d-i+2}, b_{d-j+3}, \dots, b_d\}$ satisfying $\sigma(b_{d-i+3}) < \sigma(b_{d-j+4}) < \cdots < \sigma(b_d)$, and

$$\partial_2(e(T)\otimes 1) = \sum_{j=1}^{3}(-1)^{j-1}e(T_j)\otimes x_{a_j} \in V_{(d,d,1)}\otimes_K R(-d-2) = F_1$$

for $T \in \mathrm{Tab}(d, d-1, 1, 1)$.

Theorem 2.3

If $\operatorname{char}(K) = 0$*, the complex* $\mathcal{F}_\bullet^{(d,d,1)}$ *is a minimal free resolution of* $R/I^{\mathrm{Sp}}_{(d,d,1)}$.

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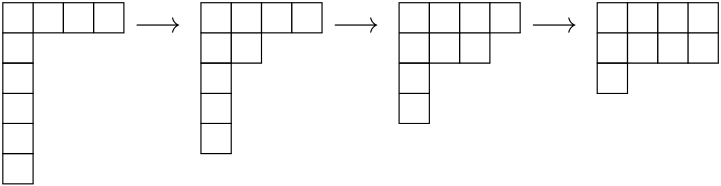
Example 2.4

We give $\mathcal{F}_\bullet^{(4,4,1)}$.

$$0 \longrightarrow R(-9)^{56} \longrightarrow R(-8)^{189} \longrightarrow R(-7)^{216} \longrightarrow R(-6)^{84} \longrightarrow R \longrightarrow R/I^{\mathrm{Sp}}_{(4,4,1)} \longrightarrow 0$$



$$
\begin{aligned}
\partial_4\left(e\left(\begin{ytableau} 1 & 2 & 3 & 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{ytableau}\right)\otimes 1\right) \quad = \quad & \left(e\left(\begin{ytableau} 5 & 2 & 3 & 4 \\ 6 & 1 \\ 7 \\ 8 \\ 9 \end{ytableau}\right) + e\left(\begin{ytableau} 5 & 3 & 2 & 4 \\ 6 & 1 \\ 7 \\ 8 \\ 9 \end{ytableau}\right) + e\left(\begin{ytableau} 5 & 4 & 2 & 3 \\ 6 & 1 \\ 7 \\ 8 \\ 9 \end{ytableau}\right) \right) \otimes x_1 \\
& -\left(e\left(\begin{ytableau} 1 & 2 & 3 & 4 \\ 6 & 5 \\ 7 \\ 8 \\ 9 \end{ytableau}\right) + e\left(\begin{ytableau} 1 & 3 & 2 & 4 \\ 6 & 5 \\ 7 \\ 8 \\ 9 \end{ytableau}\right) + e\left(\begin{ytableau} 1 & 4 & 2 & 3 \\ 6 & 5 \\ 7 \\ 8 \\ 9 \end{ytableau}\right) \right) \otimes x_5 \\
& \quad \vdots \\
& \quad \vdots \\
& -\left(e\left(\begin{ytableau} 1 & 2 & 3 & 4 \\ 5 & 9 \\ 6 \\ 7 \\ 8 \end{ytableau}\right) + e\left(\begin{ytableau} 1 & 3 & 2 & 4 \\ 5 & 9 \\ 6 \\ 7 \\ 8 \end{ytableau}\right) + e\left(\begin{ytableau} 1 & 4 & 2 & 3 \\ 5 & 9 \\ 6 \\ 7 \\ 8 \end{ytableau}\right) \right) \otimes x_9
\end{aligned}
$$

$$\partial_2\left(e\left(\ytableaushort{1234,567,8,9}\right)\otimes 1\right) = e\left(\ytableaushort{5234,8671,9}\right)\otimes x_1 - e\left(\ytableaushort{1234,8675,9}\right)\otimes x_5$$

$$+e\left(\ytableaushort{1234,5678,9}\right)\otimes x_8 - e\left(\ytableaushort{1234,5679,8}\right)\otimes x_9.$$