Edge-weighted edge ideals of very well-covered graphs

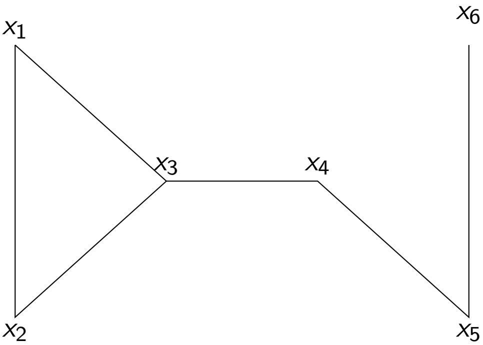
柴田 孝祐 (岡山大学)
寺井 直樹 (岡山大学)

This is a based on a joint work with S. A. S. Fakhari and S.Yassemi.
(arXiv:2003.12379)

K:a field.
$S = K[x_1, \ldots, x_n](= K[x_1, \ldots, x_h, y_1, \ldots, y_h])$: a polynomial ring.
$G = (V(G), E(G))$: a simple graph without isolated vertices, and $V(G) = \{x_1, \ldots, x_n\}$.
$I(G) := (x_i x_j \mid \{x_i, x_j\} \in E(G)) \subset S$: the edge ideal of G.

G is called unmixed if $I(G)$ is unmixed.
G is called Cohen-Macaulay(CM for short) if $S/I(G)$ is CM.

Example 1.1



$$I(G) = (x_1x_2, x_1x_3, x_2x_3, x_3x_4, x_4x_5, x_5x_6)$$

Theorem 1.2 (Gitler-Valencia, 2005)

Suppose G is an unmixed graph. Then $2\,\mathrm{ht}\,I(G) \geq \#V(G)$

Definition 1.3

Suppose G is an unmixed graph. Then G is called very well-coverd if $2\,\mathrm{ht}\,I(G) = \#V(G)$.

(*) $V(G) = X \cup Y$, $X \cap Y = \emptyset$, where $X = \{x_1, \ldots, x_h\}$ is a minimal vertex cover of G and $Y = \{y_1, \ldots, y_h\}$ is a maximal independent set of G such that $\{x_1y_1, \ldots, x_hy_h\} \subset E(G)$.

Proposition 1.4 (Morey-Reyes-Villarreal, 2008, Crupi-Rinaldo-Terai, 2011)

Let $\#V(G) = 2h$, and assume that the vertices of G are labeled such that the condition (*) is satisfied. Then G is very well-coverd if and only if the following conditions hold.

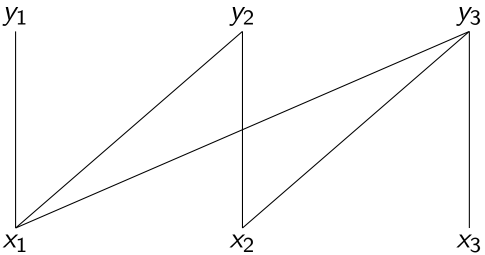
(i) *If x_iy_j, $x_jz_k \in E(G)$ then $x_iz_k \in E(G)$ for distinct i, j and k and for $z_k \in \{x_k, y_k\}$.*

(ii) *If $x_iy_j \in E(G)$ then $x_ix_j \notin E(G)$.*

(i) If $x_i y_j$, $x_j z_k \in E(G)$ then $x_i z_k \in E(G)$ for distinct i, j and k and for $z_k \in \{x_k, y_k\}$.

(ii) If $x_i y_j \in E(G)$ then $x_i x_j \notin E(G)$.

Example 1.5



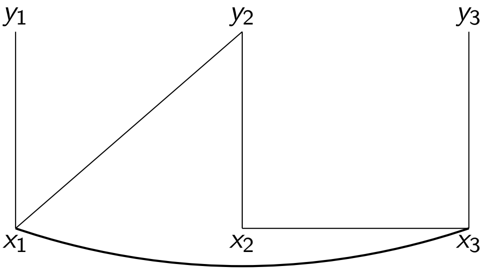
,

is very well-coverd.

(i) If $x_i y_j,\ x_j z_k \in E(G)$ then $x_i z_k \in E(G)$ for distinct i, j and k and for $z_k \in \{x_k, y_k\}$.

(ii) If $x_i y_j \in E(G)$ then $x_i x_j \notin E(G)$.

Example 1.6



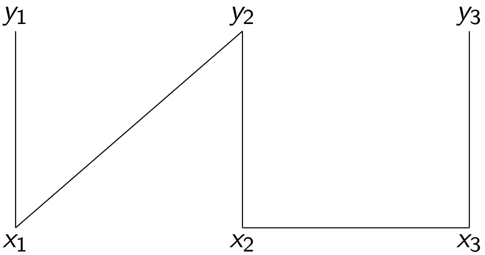
,

is very well-coverd.

(i) If x_iy_j, $x_jz_k \in E(G)$ then $x_iz_k \in E(G)$ for distinct i, j and k and for $z_k \in \{x_k, y_k\}$.

(ii) If $x_iy_j \in E(G)$ then $x_ix_j \notin E(G)$.

Example 1.7



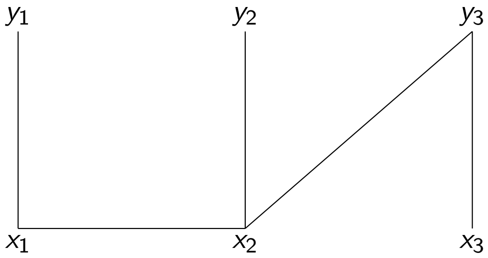
,

is not unmixed.

(i) If $x_i y_j$, $x_j z_k \in E(G)$ then $x_i z_k \in E(G)$ for distinct i, j and k and for $z_k \in \{x_k, y_k\}$.

(ii) If $x_i y_j \in E(G)$ then $x_i x_j \notin E(G)$.

Example 1.8



,

is very well-coverd.

(*) $V(G) = X \cup Y$, $X \cap Y = \emptyset$, where $X = \{x_1, \ldots, x_h\}$ is a minimal vertex cover of G and $Y = \{y_1, \ldots, y_h\}$ is a maximal independent set of G such that $\{x_1y_1, \ldots, x_hy_h\} \subset E(G)$.

Lemma 1.9

Let $\#V(G) = 2h$, and assume that the vertices of G are labeled such the condition (*) is satisfied.
If G is a CM, then there exists a suitable simultaneous change of labeling on both $\{x_i\}_{i=1}^h$ and $\{y_i\}_{i=1}^h$ such that $x_iy_j \in E(G)$ implies $i \leq j$.

(*) $V(G) = X \cup Y$, $X \cap Y = \emptyset$, where $X = \{x_1, \dots, x_h\}$ is a minimal vertex cover of G and $Y = \{y_1, \dots, y_h\}$ is a maximal independent set of G such that $\{x_1y_1, \dots, x_hy_h\} \subset E(G)$.

(**) $x_iy_j \in E(G)$ implies $i \leq j$.

Proposition 1.10 (Crupi, Rinaldo, Terai, 2011)

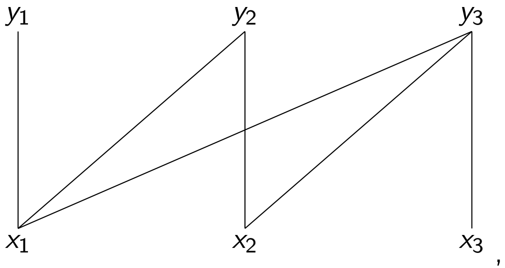
Let $\#V(G) = 2h$ and assume the conditions (*) and (**).
Then the following conditions are equivalent:

1. *G is Cohen-Macaulay;*
2. *G is unmixed;*
3. *The following conditions hold:*
   - (i) *If $x_iy_j, x_jz_k \in E(G)$ then $x_iz_k \in E(G)$ for distict i, j, k and for $z_k \in \{x_k, y_k\}$;*
   - (ii) *If $x_iy_j \in E(G)$ then $x_ix_j \notin E(G)$.*

(i) If $x_iy_j, x_jz_k \in E(G)$ then $x_iz_k \in E(G)$ for distict i, j, k and for $z_k \in \{x_k, y_k\}$;

(ii) If $x_iy_j \in E(G)$ then $x_ix_j \notin E(G)$.

Example 1.11

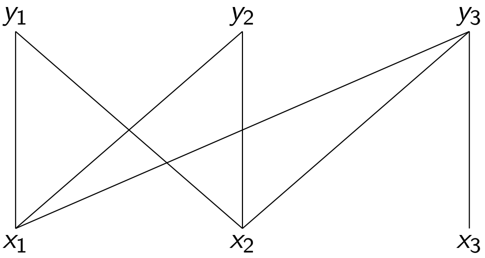


is CM very well-coverd.

(i) If $x_i y_j, x_j z_k \in E(G)$ then $x_i z_k \in E(G)$ for distict i, j, k and for $z_k \in \{x_k, y_k\}$;

(ii) If $x_i y_j \in E(G)$ then $x_i x_j \notin E(G)$.

Example 1.12



,

is unmixed and very well-coverd, but not CM .

Definition 1.13

Let $w : E(G) \longrightarrow \mathbb{Z}_{>0}$ be an edge weight on G and $G_w := (G, w)$ be an edge weighted graph.

$$I(G_w) := ((x_i x_j)^{w(x_i x_j)} \mid x_i x_j \in E(G))$$

G_w is called unmixed if $I(G_w)$ is unmixed.
G_w is called CM if $S/I(G_w)$ is CM.

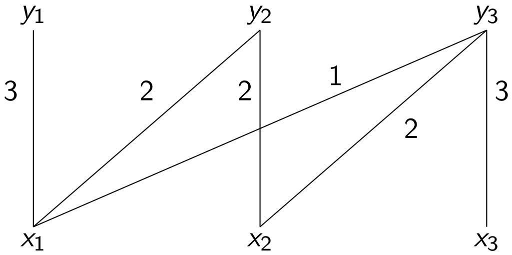
Remark 1.14

$$\sqrt{I(G_w)} = I(G)$$

$$I(G_w) \text{ is unmixed} \Longrightarrow I(G) \text{ is unmixed.}$$
$$I(G_w) \text{ is CM} \Longrightarrow I(G) \text{ is CM.}$$

But the converse is not always true.

Example 1.15



$$I(G) = (x_1y_1, x_2y_2, x_3y_3, x_1y_2, x_1y_3, x_2y_3),$$

$$I(G_w) = (x_1^3y_1^3, x_2^2y_2^2, x_3^3y_3^3, x_1^2y_2^2, x_1y_3, x_2^2y_3^2)$$

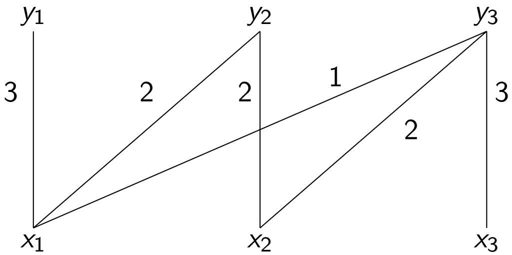
Theorem 1

Let G be a very well-covered graph with $n = 2h$ vertices, and $\operatorname{ht} I(G) = h$. We assume the condition (*). Let w is an edge weight of G. Then G_w is unmixed if and only if the following conditions hold:

(i) *If $x_i z_j \in E(G)$ then $w(x_i z_j) \leq w(x_i y_i)$ and $w(x_i z_j) \leq w(x_j y_j)$ for distinct i, j and for $z_j \in \{x_j, y_j\}$.*

(ii) *If $x_i y_j, x_j z_k \in E(G)$ then $w(x_i z_k) \leq w(x_i y_j)$ and $w(x_i z_k) \leq w(x_j z_k)$ for distinct i, j and k and for $z_k \in \{x_k, y_k\}$, or for distinct j and $i = k$ and for $z_i = y_i$.*

Example 2.1

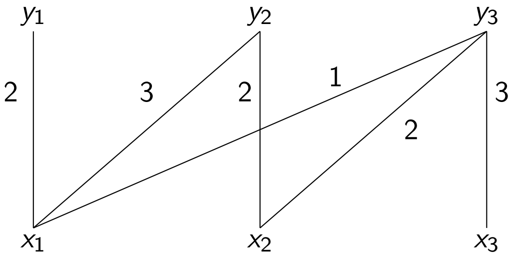


$$I(G) = (x_1y_1, x_2y_2, x_3y_3, x_1y_2, x_1y_3, x_2y_3),$$

$$I(G_w) = (x_1^3y_1^3, x_2^2y_2^2, x_3^3y_3^3, x_1^2y_2^2, x_1y_3, x_2^2y_3^2)$$

Then $I(G)$ and $I(G_w)$ are unmixed

Example 2.2



$$I(G) = (x_1y_1, x_2y_2, x_3y_3, x_1y_2, x_1y_3, x_2y_3),$$

$$I(G_w) = (x_1^2y_1^2, x_2^2y_2^2, x_3^3y_3^3, x_1^2y_2^2, x_1y_3, x_2^2y_3^2)$$

Then $I(G)$ is unmixed, but $I(G_w)$ is not unmixed.

Theorem 2

Let G be a Cohen-Macaulay very well-covered graph. Let w be an edge weight of G. Then the following conditions are equivalent:

1. *G_w is unmixed.*
2. *G_w is Cohen-Macaulay.*

Conjecture 2.3

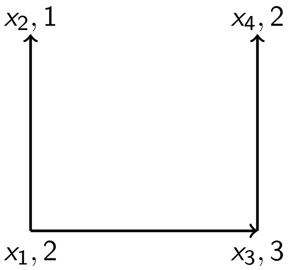
Let G be a Cohen-Macaulay very well-covered graph. Then G_w is sequentially Cohen-Macaulay.

Definition 2.4 (Pitones-Reyes-Toledo,2019)

Let $\mathcal{D} = (V(\mathcal{D}), E(\mathcal{D}))$ be an oriented graph with $V(\mathcal{D}) = \{x_1, \ldots, x_n\}$, and let $\omega : V(\mathcal{D}) \longrightarrow \mathbb{Z}_{>0}$ be a *vertex-weighted* on $\mathcal{D}$ and set $\omega_j = \omega(x_j)$. Then the *vertex-weighted edge ideal* of $\mathcal{D}$ is defined by

$$I(\mathcal{D}) = (x_i x_j^{\omega_j} \mid (x_i, x_j) \in E(\mathcal{D})).$$

Example 2.5



$$I(\mathcal{D}) = (x_1 x_2, x_1 x_3^3, x_3 x_4^2)$$

Conjecture 2.6 (Pitones-Reyes-Toledo,2019)

Let $\mathcal{D}$ be a vertex-weighted oriented graph and let G be its underlying graph. If $I(\mathcal{D})$ is unmixed and $I(G)$ is CM, then $I(\mathcal{D})$ is CM.

Example 2.7

Let $\operatorname{char}(K) = 0$ and $S = K[x_1, \ldots, x_{11}]$.

$$\begin{aligned} I(G) =& (x_1x_3, x_1x_4, x_1x_7, x_1x_{10}, x_1x_{11}, x_2x_4, x_2x_5, x_2x_8, x_2x_{10}, x_2x_{11}, \\ & x_3x_5, x_3x_6, x_3x_8, x_3x_{11}, x_4x_6, x_4x_9, x_4x_{11}, x_5x_7, x_5x_9, x_5x_{11}, \\ & x_6x_8, x_6x_9, x_7x_9, x_7x_{10}, x_8x_{10}). \end{aligned}$$

This ideal comes from the triangulation of the real projective plane.

$$\begin{aligned} I(\mathcal{D}_1) =& (x_1x_3, x_1x_4, x_1x_7, x_1x_{10}, x_1x_{11}^2, x_2x_4, x_2x_5, x_2x_8, x_2x_{10}, x_2x_{11}^2, \\ & x_3x_5, x_3x_6, x_3x_8, x_3x_{11}^2, x_4x_6, x_4x_9, x_4x_{11}^2, x_5x_7, x_5x_9, x_5x_{11}, \\ & x_6x_8, x_6x_9, x_7x_9, x_7x_{10}, x_8x_{10}). \end{aligned}$$

$$\begin{aligned} I(\mathcal{D}_2) =& (x_1x_3, x_1x_4, x_1x_7, x_1x_{10}, x_1x_{11}, x_2x_4, x_2x_5, x_2x_8, x_2x_{10}, x_2x_{11}, \\ & x_3x_5, x_3x_6, x_3x_8, x_3x_{11}, x_4x_6, x_4x_9, x_4x_{11}, x_5x_7, x_5x_9, x_5x_{11}, \\ & x_6x_8, x_6x_9, x_7^2x_9, x_7x_{10}, x_8x_{10}). \end{aligned}$$

Then $I(G)$ is CM. However, *Macaulay2* computation shows that $I(\mathcal{D}_1)$ is unmixed and it satisfies (S_2) condition, but is not CM, and $I(\mathcal{D}_2)$ is unmixed, but does not satisfy (S_2) condition.

Example 2.8

Let $\mathrm{char}(K) = 0$ and $S = K[x_1, \ldots, x_{11}]$.

$$\begin{aligned} I(G_{w_1}) =(& x_1x_3, x_1x_4, x_1x_7, x_1x_{10}, x_1x_{11}, x_2x_4, x_2x_5, x_2x_8, x_2x_{10}, x_2x_{11}, \\ & x_3x_5, x_3x_6, x_3x_8, x_3x_{11}, x_4x_6, x_4x_9, x_4x_{11}, x_5x_7, x_5x_9, x_5x_{11}, \\ & x_6x_8, x_6x_9, x_7x_9, x_7x_{10}, x_8^2x_{10}^2). \end{aligned}$$

$$\begin{aligned} I(G_{w_2}) =(& x_1^2x_3^2, x_1^2x_4^2, x_1^2x_7^2, x_1^2x_{10}^2, x_1^2x_{11}^2, x_2^2x_4^2, x_2^2x_5^2, x_2^2x_8^2, x_2^2x_{10}^2, x_2^2x_{11}^2, \\ & x_3^2x_5^2, x_3^2x_6^2, x_3^2x_8^2, x_3^2x_{11}^2, x_4^2x_6^2, x_4^2x_9^2, x_4^2x_{11}^2, x_5^2x_7^2, x_5^2x_9^2, x_5^2x_{11}^2, \\ & x_6^2x_8^2, x_6^2x_9^2, x_7^2x_9^2, x_7^2x_{10}^2, x_8x_{10}). \end{aligned}$$

Then $I(G)$ is Cohen-Macaulay.
However, *Macaulay2* computation shows that $I(G_{w_1})$ is unmixed, but does not satisfy (S_2) condition.
On the other hand, *Macaulay2* computation shows that $I(G_{w_2})$ is unmixed and it satisfies (S_2) condition, but it is not Cohen-Macaulay.