Cameron-Walker graph の regularity, h-多項式, 次元, 深度について

菅野裕樹

Department of Pure and Applied Mathematics, Graduate School of Information Science and Technology, Osaka University

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Preliminaries

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1. $G = (V(G), E(G))$: Graph.
2. $\mathrm{m}(G)$: Matching number of G.
3. $\mathrm{im}(G)$: Induced matching number of G. In particular, we say G is *gap-free* if $\mathrm{im}(G) = 1$.
4. S : Independent set.
5. $I(G)$: Edge ideal of G.
6. $\mathrm{reg}(G)$, $\dim(G)$, $\mathrm{depth}(G)$, $\deg(G)$: Regularity, dimension, depth, degree of h-polynomial of $R/I(G)$.

Preliminaries

Cameron-Walker graph

A finite connected simple graph G is *Cameron–Walker* if $\mathrm{im}(G) = \mathrm{m}(G)$ and if G is neither a star graph nor a star triangle.

Figure of Cameron-Walker graph

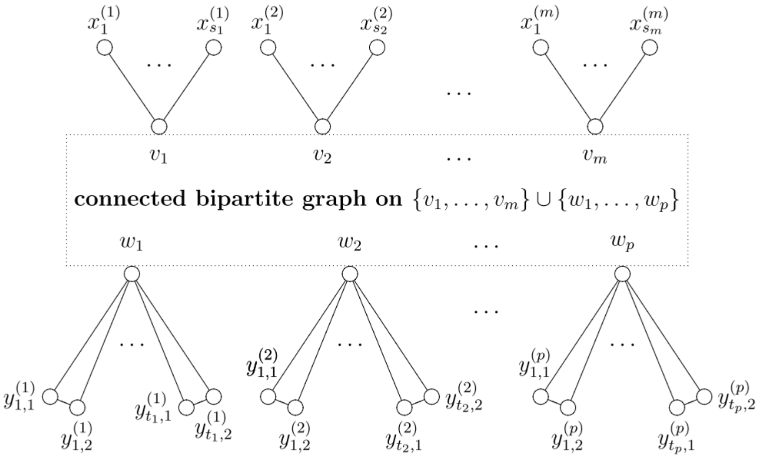

Figure: Cameron-Walker graph

Known results

[Hà and Van Tuyl(2008)], [Katzman(2006)]

For any graph G, following holds:

$$\mathrm{im}(G) \leq \mathrm{reg}(G) \leq \mathrm{m}(G).$$

[Trung(2020)]

Let G be a graph. Then, $\mathrm{reg}(G) = \mathrm{m}(G)$ if and only if each connected component of G is either a pentagon or a Cameron-Walker graph or a star graph or a star triangle.

Known results

[W. V. Vasconcelos(1998)]

Let G be a graph. Then

$$\deg(G) - \operatorname{reg}(G) \le \dim(G) - \operatorname{depth}(G) \cdots (*).$$

The equality holds if and only if $\beta_{p,p+r}(R/I(G)) \neq 0$. In particular, if $R/I(G)$ is Cohen-Macaulay, then $\deg(G) = \operatorname{reg}(G)$.

[Hibi, Matsuda, Van Tyul(2019)]

Let G be a graph on n vertices. Then

$$\deg(G) + \operatorname{reg}(G) \le n.$$

Problem

Problem (#)

For fixed n, determine

$$G(n) = \{(h, r, d, e) \in \mathbb{N}^4 : \exists G \text{ s.t. } |V(G)| = n, \deg(G) = h, \operatorname{reg}(G) = r, \dim(G) = d, \operatorname{depth}(G) = e\}.$$

Main result 1

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For $n \geq 3$, let

$$C^{-}(n) := \left\{(a,b) \,\middle|\, a \leq b, 1 \leq a \leq \left\lfloor \frac{n}{2} \right\rfloor, 1 \leq b \leq n-2 \right\} \subseteq \mathbb{N}^2$$

and

$$C^{+}(n) := \{(a,b) \mid 1 \leq a \leq b \leq n-1\} \subseteq \mathbb{N}^2.$$

For all $n \geq 3$, $C^{-}(n) \subseteq \mathrm{Graph}_{\mathrm{depth,dim}}(n) \subseteq C^{+}(n)$.

Main result 1

HKKMVT2020

For all $n \geq 2$, we have $\mathrm{Graph}_{\mathrm{depth,dim}}(n) \subseteq \mathrm{Graph}_{\mathrm{depth,dim}}(n+1)$.

Let a, b be integers with $1 \leq a \leq b$. Then $(a, b) \in \mathrm{Graph}_{\mathrm{depth,dim}}(a+b)$.

Let $1 \leq a \leq b$ and $n \geq 2$ be integers. Assume that $b \leq n-1$. If $a \leq b + 1 - \left\lceil \frac{b}{n-b} \right\rceil$, then $(a, b) \in \mathrm{Graph}_{\mathrm{depth,dim}}(n)$.

Let $n \geq 6$ be an integer. If $\left\lceil \frac{n}{2} \right\rceil + 1 \leq b \leq n-2$, then

$$b + 1 - \left\lceil \frac{b}{n-b} \right\rceil \geq \left\lfloor \frac{n}{2} \right\rfloor$$

holds.

Main result 2

HKKMVT2020

Let $n \geq 5$ be an integer. Then

(i) If n is even, then

$$
\begin{aligned}
& CW_{\text{depth,reg,dim,deg}\,h}(n) \\
= \;& \left\{(a,d,d,d) \in \mathbb{N}^4 \,\middle|\, 3 \leq a \leq d \leq \left\lfloor \frac{n-1}{2} \right\rfloor, n < a + 2d \right\} \\
\cup \;& \left\{(a,a,d,d) \in \mathbb{N}^4 \,\middle|\, 3 \leq a < d \leq n - a, n \leq 2a + d - 1 \right\} \\
\cup \;& \left\{(a,r,d,d) \in \mathbb{N}^4 \,\middle|\, \begin{array}{c} 3 \leq a < r < d < n - r, \\ n + 2 \leq a + r + d \end{array} \right\} \\
\cup \;& \{(2,2,n-2,n-2),(2,2,n-3,n-3)\}.
\end{aligned}
$$

Main result2

(ii) If n is odd, then

$$
\begin{aligned}
& CW_{\text{depth,reg,dim,deg}\,h}(n) \\
= \;& \left\{(a,d,d,d) \in \mathbb{N}^4 \;\middle|\; 3 \le a \le d \le \left\lfloor \frac{n-1}{2} \right\rfloor, n < a + 2d \right\} \\
\cup \;& \left\{(a,a,d,d) \in \mathbb{N}^4 \;\middle|\; 3 \le a < d \le n - a, n \le 2a + d - 1 \right\} \\
\cup \;& \left\{(a,r,d,d) \in \mathbb{N}^4 \;\middle|\; \begin{array}{c} 3 \le a < r < d < n - r, \\ n + 2 \le a + r + d \end{array} \right\} \\
\cup \;& \{(2,2,n-2,n-2),(2,2,n-3,n-3)\} \\
\cup \;& \left\{\left(2, \frac{n-1}{2}, \frac{n-1}{2}, \frac{n-1}{2}\right)\right\}.
\end{aligned}
$$

Main result2

[Hibi, Kimura, Matsuda, Tsuchiya(2019)]

For any Cameron-Walker graph,

$$\deg(G) = \dim(G) = \sum_{i=1}^{m} s_i + \sum_{j=1}^{p} \max\{t_j, 1\}.$$

And

$$\mathrm{reg}(G) = m + \sum_{i=1}^{p} t_j.$$

For any Cameron-Walker graph, The equality (*) holds if and only if

$$\sum_{i=1}^{m} s_i + n \geq \sum_{j=1}^{p} t_j + m.$$

Main result2

[Hibi, Kimura, Matsuda, Tsuchiya(2019)]

There is no Cameron-Walker graph G on n vertices with $\mathrm{depth}(R/I(G)) = 1$.

[HKKMVT2020]

Assume that $n \geq 5$ is even (resp. odd). Then $(2, b) \in CW_{\mathrm{depth},\dim}(n)$ if and only if $b = n - 3$ or $b = n - 2$ (resp. $b = n - 3$ or $b = n - 2$ or $b = \frac{n-1}{2}$).

Main result2

[HKKMVT(2020)]

Let G be a Cameron-Walker graph with notation Then
$\operatorname{depth}(G) = \min_{V \subset \{v_1, \ldots, v_m\}} \{f(V)\}$,
where $f(V) = \sum_{v_i \in V} s_i + m - |V| + \sum_{N_{G_{\text{bip}}}(w_j) \not\subset V} t_j + \left| \left\{ j : N_{G_{\text{bip}}}(w_j) \subset V \right\} \right|$.

Main result2

[HKKMVT(2020)]

Let G be a Cameron-Walker graph with notation as in Figure **??**. Then

(i) the inequality

$$m + |\{j : t_j > 0\}| \leq \operatorname{depth}(R/I(G)) \leq \sum_{j=1}^{p} t_j + m = \operatorname{reg}(R/I(G))$$

holds. Moreover, if the bipartite part of G is the complete bipartite graph, then

$$\operatorname{depth}(R/I(G)) = \min\left\{\sum_{i=1}^{m} s_i + p, \sum_{j=1}^{p} t_j + m\right\}.$$

(ii) $2 \leq \operatorname{depth}(R/I(G)) \leq \left\lfloor \frac{|V(G)|-1}{2} \right\rfloor$.

Main result2

[HKKMVT(2020)]

(*iii*) $\operatorname{depth}(R/I(G)) + \dim R/I(G) \leq |V(G)|$.

(*iv*) $|V(G)| < \operatorname{depth}(R/I(G)) + 2\dim R/I(G)$.

(*v*) $|V(G)| = 2m + 3p$ and $\dim R/I(G) = \operatorname{depth}(R/I(G)) = m + p$ if $R/I(G)$ is Cohen-Macaulay.

[HKKMVT2020]

Let G be a Cameron-Walker graph. Then

(*i*) $|V(G)| + 1 \leq \operatorname{depth}(G) + \operatorname{reg}(G) + \dim(G)$.

(*ii*) Assume that $\operatorname{depth}(G) < \operatorname{reg}(G)$. If $|V(G)| + 1 = \operatorname{depth}(G) + \operatorname{reg}(G) + \dim(G)$, then $\operatorname{reg}(G) = \dim(G)$.

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