F-PURE SINGULARITIES IN EQUAL CHARACTERISRIC ZERO

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Throughout this talk R denotes a commutative Noetherian ring with unity. Let S be an R-algebra. The morphism $R \to S$ is said to be *pure* if for all R-module M, $M \to M \otimes_R S$ is injective. If R is a direct summand of S, then $R \to S$ is pure. For example, pure morphisms appear when we consider quotients Y of varieties X by actions of linearly reductive groups. We can expect that singularities of X and Y are closely related. Hence, the following is a natural question.

Question 1. *Suppose that R and S are essentially of finite type over $\mathbb{C}$, and $R \to S$ is pure. If S has some property, then so does R?*

Remark 1. Pure homomorphisms are not necessarily preserved under reductions modulo $p > 0$. Hence, Question 1 can not be solved by simply using reductions modulo $p > 0$.

Regarding Question 1, the following are known:

(1) If S has rational singularities, then R has rational singularities [1].
(2) If S is of klt type, then R is of klt type [6]. When R and S are $\mathbb{Q}$-Gorenstein, Schoutens showed the result in [5] via ultraproducts.
(3) If S has Du Bois singularities, then R has Du Bois singularities [2].

To state the main question of this talk, we need to introduce some classes of singularities.

Definition 1. A ring R of characteristic $p > 0$ is said to be *F-pure* if the Frobenius morphism $F : R \to R$ is pure.

For F-pure singularities, Question 1 is clear by definition.

Definition 2. Let X be a $\mathbb{Q}$-Gorenstein normal integral scheme essentially of finite type over $\mathbb{C}$. Let $f : Y \to X$ be a log resolution of X. Suppose that $K_Y = f^*K_X + a_{E_i}E_i$, where E_i are distinct exceptional divisors. We say that X has *log canonical singularities* if $a_i \geqslant -1$ for all i.

Log canonical singularities are important in birational geometry and a wider class than log terminal singularities. They are expected to be equivalent to singularities of dense F-pure type, which are defined via modulo $p > 0$ reductions. Roughly speaking, dense F-purity means that the reduction modulo $p > 0$ is F-pure for infinitely many p. If R is of dense F-pure type, then R has log canonical singularities [3]. The converse is an open problem.

Since Question 1 has an affirmative answer regarding log terminal singularities, we expect that the following holds.

Question 2 (cf. [6, Question 2.11])**.** *If R and S are essentially of finite type over $\mathbb{C}$, $R \to S$ is pure, and S has log canonical singularities, then does R have log canonical singularities?*

However, reductions modulo $p > 0$ of log canonical singularities are difficult to treat since the equivalence of dense F-purity and log canonicity have not been shown. Hence, we consider the following question, which is equivalent to the above one under some conjecture.

Question 3. *With notation as above, if S is of dense F-pure type, then is R of dense F-pure type?*

To show this, we introduce ultra-F-pure singularities analogous to ultra-F-regular singularities introduced by Schoutens [5].

Theorem 1. *Suppose that $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$ are normal local domains essentially of finite type over $\mathbb{C}$, $R \to S$ is pure, R is $\mathbb{Q}$-Gorenstein, and S is of dense F-pure type. Then R is of dense F-pure type.*

We can construct a variant of perfect closure in equal characteristic zero via ultraproducts. Using this algebra, the theorem can be shown as in characteristic $p > 0$. For the proof, it is essential to compare local cohomologies in equal characteristic zero with ones in positive characteristic. To do this, we need to utilize the theory of p-standard sequences introduced in [4], which are a generalization of regular sequneces and d^+-sequences.

References

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