ON THE CATEGORY OF COFINITE MODULES AND LOCAL COHOMOLOGY MODULES

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This is a joint work with Ryo Takahashi [5]. Throughout this abstruct, R is a Noetherian ring, I is an ideal of R, and, $\mathsf{Mod}\,R$ is a category of R-modules. An *I-cofinite* R-module is by definition an R-module X that satisfies both of the following two conditions (a) and (b).

(a) $\operatorname{Supp} X$ is contained in $\mathrm{V}(I)$.

(b) $\operatorname{Ext}_R^i(R/I, X)$ is finitely generated for all integers i.

Hartshorne [4] introduced the notion of an I-cofinite module, and constructed an example where a local cohomology module $\mathrm{H}_I^i(M)$ is not I-cofinite, which is a counterexample to a conjecture of Grothendieck [3]. Since then, so many people have worked on the question asking when $\mathrm{H}_I^i(M)$ is I-cofinite, and so many results on it have been obtained; see for example [2] and references therein.

Denote by $\mathsf{Cof}_I(R)$ the full subcategory of $\mathsf{Mod}\,R$ consisting of I-cofinite R-modules. After proving results on the relationship between the categorical structure of $\mathsf{Cof}_I(R)$ and the cofiniteness of local cohomology modules, Bahmanpour [2] posed the following question.

Question 0.1 (Bahmanpour)**.** Suppose that $\mathsf{Cof}_I(R)$ is an abelian subcategory of $\mathsf{Mod}\,R$. Is then $\mathrm{H}_I^i(M)$ an I-cofinite R-module for all finitely generated R-modules M and all integers i ?

The purpose of this talk is to provide a couple of answers to Question mainly by means of techniques of subcategories of modules. Denote by $\mathsf{Cof}_I^0(R)$ the full subcategory of $\mathsf{Mod}\,R$ consisting of R-modules X satisfying the above condition (b) only; such modules are called *I-ETH-cofinite* and investigated, see [1] for example. Note that for an R-module M and an integer i there are equivalences

$$\mathrm{H}_I^i(M) \text{ is } I\text{-cofinite} \iff \mathrm{H}_I^i(M) \in \mathsf{Cof}_I(R) \iff \mathrm{H}_I^i(M) \in \mathsf{Cof}_I^0(R).$$

The main result of this talk is the following theorem.

Theorem 0.2. *Assume that one of the following three conditions is satisfied.*

(1) $\mathsf{Cof}_I^0(R)$ *is abelian.*

(2) $\mathsf{Cof}_I(R)$ *is Serre, and* $\mathrm{H}_I^i(R)$ *is I-cofinite for any integer i.*

(3) $\mathsf{Cof}_I(R)$ *is abelian,* $\mathrm{H}_I^i(R)$ *is I-cofinite for any integer i, and* $\operatorname{Sing} R$ *is contained in* $\mathrm{V}(I)$.

Then $\mathrm{H}_I^i(M)$ *is I-cofinite for any finitely generated R-module M and any integer i.*

References

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