RECENT PROGRESS IN THE THEORY OF DUAL F-SIGNATURE

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Abstract

Introduction. This talk concerns local rings of positive characteristic p. In this world, the Frobenius endomorphism, a map sending $x \mapsto x^p$, is used to study singularities. A popular approach is to define classes of singularities or invariants of singularities via the properties of Frobenius. Two such classes are *F-rational* and *(strongly) F-regular* singularities.

In order to state definitions, let me introduce a notation. For an R-module M, I will denote by $F^e_* M$ the R-module structure on M pushed by the eth iterate of Frobenius: $R \xrightarrow{x \mapsto x^{p^e}} R \to M$. Thus $aF^e_* M = F^e_*(a^{p^e} M)$. I will now assume that R is a local domain with the maximal ideal $\mathfrak{m}$ and that $F^e_* R$ is a finitely generated R-module, this is almost equivalent to the residue field k being such that $\dim_k k^{1/p} < \infty$. Then, under mild assumptions, F-regularity asserts that for every $c \neq 0$ there is an e and a map $\phi \colon F^e_* R \to R$ such that $\phi(F^e_* c) = 1$, while F-rationality demands a map on the dualizing module ω_R of a Cohen-Macaulay ring, $\psi \colon F^e_* \omega_R \to \omega_R$ need to be such that $\psi(F^e_* c\omega_R) = \omega_R$. I refer to the survey [TW18, TW14] for some background on these classes of singularities.

As for the invariants, this work originates in the theory of *F-signature*. As my space is restricted, I will summarize only the following fundamental properties.

I Convergence in the definition: $\mathrm{s}(R) = \lim_{e\to\infty} \frac{\max\{N | F^e_* R \to \oplus^N R \to 0\}}{\text{generic rank of } F^e_* R} \in [0, 1]$.

II Detection of *F-regular* singularities: $\mathrm{s}(R) > 0$ if and only if R is F-regular.

III Detection of singularities: $\mathrm{s}(R) = 1$ if and only if R is regular.

IV Lower semicontinuity: $\{\mathfrak{p} \in \operatorname{Spec} R \mid \mathrm{s}(R_\mathfrak{p}) > a\}$ is open for all $a \in \mathbb{R}$.

V Connection with Hilbert–Kunz multiplicity: $\mathrm{s}(R) = \inf\{\mathrm{e_{HK}}(I) - \mathrm{e_{HK}}(J) \mid I \subsetneq J, \sqrt{I} = \mathfrak{m}\}$,

where the *Hilbert–Kunz multiplicity* of an $\mathfrak{m}$-primary ideal I can be defined as

$$\mathrm{e_{HK}}(I) = \lim_{e\to\infty} \frac{\text{length of } R/I \otimes_R F^e_* R}{\operatorname{rank} F^e_* R}.$$

I refer to the survey [Hun13] for more information on F-signature and Hilbert–Kunz multiplicity.

In part due to the connection with F-regularity, F-signature has important applications in algebraic geometry and, perhaps, it is now the most important of all Frobenius-born invariants. *Dual F-signature* was introduced by Sannai in [San15] with the purpose to similarly measure *F-rational* singularities. Like the given definition of F-rationality, it involves the dualizing module ω_R:

$$\mathrm{s_{dual}}(R) = \limsup_{e\to\infty} \frac{\max\{N \mid \text{ there is } F^e_* \omega_R \to \oplus^N \omega_R \to 0\}}{\text{generic rank of } F^e_* \omega_R}.$$

Results. I will report on a joint work with Tucker [ST] which establishes the analogues of the five fundamental properties (and more) for the dual F-signature, thus completing the envisioned parallel between the two theories. From Sannai's work it was known that dual F-signature satisfies (II), (III), and a weak form of (IV): $\mathrm{s_{dual}}(R_\mathfrak{p}) \geq \mathrm{s_{dual}}(R)$ as the definition localizes.

For our work the crucial property is (V). Previously, in [HY22][1] Hochster and Yao have detected F-rationality using a relative Hilbert–Kunz invariant

$$\inf\{e_{\mathrm{HK}}(\underline{x}) - e_{\mathrm{HK}}(J) \mid \underline{x} \subsetneq J, \underline{x} \text{ is generated by a system of parameters}\}.$$

Unfortunately, this invariant is not equal to dual F-signature (for example, Veronese subrings of $k[x, y]$ of degree at least 3) and does not detect regular rings (for example, the second degree Veronese of $k[x, y, z]$). In an earlier version of our work (posted in 2019) we have found how to correct the Hochster–Yao definition: we detect non-singularity by taking instead

$$\inf\left\{\frac{e_{\mathrm{HK}}(\underline{x}) - e_{\mathrm{HK}}(J)}{\ell(R/\underline{x}) - \ell(R/J)} \mid \underline{x} \subsetneq J, \underline{x} \text{ is generated by a system of parameters}\right\}. \tag{1}$$

Later came a breakthrough and we are now able to show that the infimum in (1) is equal to the dual F-signature, thus settling (V). This equality allows to establish (IV) and (I) using the *uniform convergence techniques* of the Hilbert–Kunz theory.

Thus the theories of dual F-signature and F-signature are similar and this may lead to applications in a broader class of F-rational singularities. However, there are important differences. One example is that there is no known formula for dual F-signature of quotient singularities. On the other hand, the infimum in (1) is in fact a minimum, while the corresponding statement for F-signature would imply the notorious conjecture on equivalence of strong and weak F-regularity. We prove that the minimum exists by showing that $\underline{x}$ can be fixed and J can be restricted to socle ideals $J \subseteq \underline{x} : \mathfrak{m}$ and then we show that as a function of J the fraction in (1) is lower semicontinuous on the Grassmannian of the socle $\underline{x} : \mathfrak{m}/\underline{x}$.

The linear algebra theorem. The key to proving that $s_{\mathrm{dual}}(R)$ is equal to the infimum in (1) is a theorem in linear algebra. Let V be a finite-dimensional vector space over a field k. We prove that there is a constant C (which depends only on $\dim_k V$) with the following property. For any finite dimensional k-vector space X and a subspace $H \subseteq \mathrm{Hom}(V, X)$ if

$$\alpha = \inf\left\{\frac{\dim\left(\sum_{h\in H} h(U)\right)}{\dim U} \;\middle|\; 0 \neq U \subseteq V\right\}$$

then there exist an injection $\oplus^{\lfloor \alpha \rfloor - C} V \to X$. If you have seen a statement like this before, please let me know.

References

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[1]This paper was published only recently, but, in fact, its preprint was already used in Sannai's proof of (II).