On p-adic deformation of cohomology theory and singularities

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This is a report on papers [1] and [2], which has been worked out jointly with E. Tavanfar (IPM). Our aim is to talk about problems on Cohen-Macaulay rings/modules in different perspectives, related p-adic cohomology theory and an open question on the characterization of Cohen-Macaulay property in terms of crystalline/rigid cohomology. We are motivated by the following conjecture.

Conjecture 1 (Hochster). *Every complete local domain* $(R, \mathfrak{m})$ *has a small Cohen-Macaulay module.*

In contrast to this, we already have the following result.

Theorem 2 (André, Bhatt, Hochster-Huneke). *Every Noetherian local ring has a big Cohen-Macaulay algebra. In particular, if* R *is a complete local domain with residue field characteristic* $p > 0$. *Then the absolute integral closure* R^+ *is a big Cohen-Macaulay algebra.*

The following question arises in the course of studying the absolute integral closures.

Question 3. *Let* $(R, \mathfrak{m})$ *be a Noetherian local domain. Let* R^+ *be the absolute integral closure. Then what can we say about the structure between* R *and* R^+

Theorem 4 (Shimomoto,Tavanfar; [2]). *Let* $(R, \mathfrak{m})$ *be a complete local domain of dimension* 2 *of equal characteristic zero. Then* R^+ *is a direct limit of normal Gorenstein local sub-algebras of* R^+.

As to Conjecture 1, we have the following positive result.

Theorem 5 (Shimomoto-Tavanfar; [1]). *Let* $(R, \mathfrak{m}, k)$ *be a* d*-dimensional quasi-Gorenstein local ring which is a homomorphic image of a Gorenstien ring. Assume that* $\underline{x} := x_1, \ldots, x_{d-3}$ *be a regular sequence such that* $R/(\underline{x})R$ *is quasi-Gorenstein and Buchsbaum and* $H^2_{\mathfrak{m}}(R/(\underline{x})R) = k$. *Then* R *has a small Cohen-Macaulay module.*

Concretely, there is a special element $x \in \mathfrak{m}$ *such that if* Ω_R *is the first syzygy in the minimal free resolution of* $\omega_{R/xR}$ *as an* R*-module, then* Ω_R *is a small Cohen-Macaulay module of rank* 2.

References

[1] K. Shimomoto and E. Tavanfar, *Remarks on the small Cohen-Macaulay conjecture and new instances of maximal Cohen-Macaulay modules*, J. Algebra **634** (2023), 667–697.

[2] K. Shimomoto and E. Tavanfar, *On local rings without small Cohen-Macaulay algebras in mixed characteristic*, https://arxiv.org/abs/2109.12700.