THE KOSZUL PROPERTY AND FREE RESOLUTIONS OVER G-STRETCHED LOCAL RINGS

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Let $(R, \mathfrak{m}, k)$ be a noetherian local ring or a standard graded k-algebra with the unique (graded) maximal ideal $\mathfrak{m}$. In the case R is artinian local, with $e(R), \operatorname{embdim}(R) = \dim_k \mathfrak{m}/\mathfrak{m}^2$ being the multiplicity, respectively the embedding dimension of R, it is not hard to show that the socle degree of R is at most $e(R) - \operatorname{embdim}(R)$. Sally initiated the study of artinian local rings $(R, \mathfrak{m})$ such that its socle degree exactly equals $e(R) - \operatorname{embdim}(R)$. She called such an artinian ring a *stretched* ring. It is not hard to show that an artinian ring is stretched if and only if $\mathfrak{m}^2$ is a principal ideal. It is natural to ask what happens for a non-artinian ring R if $\mathfrak{m}^2$ is a principal ideal. We call such rings *g-stretched*, or *generalized stretched* ring. N. Matsuoka inquired about the structure of g-stretched rings.

Question 1 (Matsuoka)**.** Let $(S, \mathfrak{m})$ be a regular local ring of dimension $d \geq 2$, and $I \subseteq \mathfrak{m}^2$ an ideal such that the following conditions hold:

(1) $\mathfrak{m}^2/I$ is a cyclic module,
(2) $\dim(S/I) = 1$ and $\operatorname{depth}(S/I) = 0$.

Is it true that $I = Q\mathfrak{m}$, where Q is generated by a regular sequence of $d-1$ elements in $\mathfrak{m} \setminus \mathfrak{m}^2$?

Our first main result answers in the positive this question.

Next we discuss the Koszul property and free resolutions over g-stretched local rings. Herzog and Iyengar [6] defined for a noetherian local ring $(R, \mathfrak{m}, k)$ and a finitely generated R-module M, the *linearity defect* $\operatorname{ld}_R M$ of M. They defined such a local ring to be Koszul if $\operatorname{ld}_R k = 0$. It is well-known that $(R, \mathfrak{m})$ is a Koszul local ring if and only if $\operatorname{gr}_{\mathfrak{m}} R$ is a standard graded Koszul k-algebra.

The result below extends a result due to Avramov, Iyengar and Şega [1, Theorem 4.1] who treated the case R is g-stretched and $\mathfrak{m}^3 = 0$.

Theorem 2. *Let $(S, \mathfrak{n}, k)$ be a regular local ring, and $I \subseteq \mathfrak{n}^2$ an ideal. Denote $R = S/I$ and $\mathfrak{m} = \mathfrak{n}/I$. Assume that R is a g-stretched ring. Then the following are equivalent:*

(1) *R is Koszul;*
(2) *Either $\dim R = 1$, or R is artinian, $\mathfrak{m}^3 = 0$, and $\operatorname{rank}_k(0 : \mathfrak{m}) \leq \mu(\mathfrak{m}) - 1$ unless $\mathfrak{m}^2 = 0$.*

By work of Avramov, Eisenbud, and Peeva, it is well-known that if $(R, \mathfrak{m})$ is a standard graded k-algebra, then R is a Koszul algebra if and only if $\operatorname{reg}_R k < \infty$. The following result was proved in [6].

Theorem 3 (Herzog–Iyengar). *Let $(R, \mathfrak{m})$ be a standard graded k-algebra. Then the following are equivalent:*

(1) *R is a Koszul algebra;*
(2) $\operatorname{ld}_R k = 0$*;*
(3) $\operatorname{ld}_R k < \infty$*.*

However, the analogous statement of (3) $\Longrightarrow$ (2) for local rings, i.e. for a local ring $(R, \mathfrak{m}, k)$, that $\operatorname{ld}_R k < \infty$ implies R being Koszul, remains open. The following result gives an affirmative answer to this question when $(R, \mathfrak{m}, k)$ is a g-stretched quotient of a regular local ring such that $\operatorname{char} k = 0$.

Theorem 4. *Let $(S, \mathfrak{n}, k)$ be a regular local ring such that* $\operatorname{char} k = 0$*, and $I \subseteq \mathfrak{n}^2$ an ideal. Denote $R = S/I$ and $\mathfrak{m} = \mathfrak{n}/I$. Assume that R is a g-stretched ring. Then R is Koszul if and only if* $\operatorname{ld}_R k < \infty$*.*

This is from joint work with Do Van Kien.

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