GOVOROV-LAZARD TYPE THEOREMS AND COMPLETE BIG COHEN-MACAULAY MODULES

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This talk is based on [4]. Let A be a ring and denote by $\mathsf{Mod}\,A$ (resp. $\mathsf{mod}\,A$) the category of (right) A-modules (resp. finitely presented A-modules). Let $\mathcal{C}$ be an additive subcategory of $\mathsf{Mod}\,A$ closed under direct limits. In general, it is a delicate problem whether every module in $\mathcal{C}$ can be presented as a direct limit of modules in $\mathcal{C} \cap \mathsf{mod}\,A$. If this is possible, we say that a *Govorov–Lazard type theorem* holds for $\mathcal{C}$, and write

$$\varinjlim(\mathcal{C} \cap \mathsf{mod}\,A) = \mathcal{C}.$$

For example, we have $\varinjlim \mathsf{mod}\,A = \mathsf{Mod}\,A$ and $\varinjlim \mathsf{proj}\,A = \mathsf{Flat}\,A$, where $\mathsf{Flat}\,A$ (resp. $\mathsf{proj}\,A$) denotes the category of flat (resp. finitely generated projective) A-modules. The second equality is due to Govorov (1965) and Lazard (1969). Holm–Jørgensen [2] proved that a Govorov–Lazard type theorem may not hold for the category of Gorenstein-flat modules.

Let R be a commutative noetherian local ring. An R-module M is called *(balanced) big Cohen–Macaulay* if every system of parameters of R is an M-regular sequence. We call an R-module M a *weak* big Cohen–Macaulay if every system of parameters of R is a weak M-regular sequence. We denote by $\mathsf{WCM}\,R$ the category of weak big CM modules. Then

$$\mathsf{WCM}\,R \cap \mathsf{mod}\,R = \mathsf{CM}\,R,$$

where the right-hand side denotes the category of (maximal) Cohen–Macaulay modules. Holm [1] proved that $\varinjlim \mathsf{CM}\,R = \mathsf{WCM}\,R$ holds for every CM (=Cohen–Macaulay) local ring R with a canonical module. Our first result extends this to orders over a CM local ring R with a canonical module. Recall that a module-finite R-algebra A is called an *R-order* if A is CM as an R-module. We denote by $\mathsf{CM}\,A$ (resp. $\mathsf{WCM}\,A$) the category of A-modules being CM (resp. weak big CM) as R-modules.

Theorem 1. *Let R be a CM local ring with a canonical module and let A be an R-order. Then $\varinjlim \mathsf{CM}\,A = \mathsf{WCM}\,A$.*

It is well known that every pure-injective module over a finite-dimensional algebra is a direct summand of a direct product of finitely presented modules. By Theorem 1, we can generalize this fact to complete orders.

Corollary 1. *Let R and A be as above and assume R is complete. Then every pure-injective complete big CM module is a direct summand of a direct product of CM modules.*

If R is not CM, there would be little hope that we could have $\varinjlim \mathsf{CM}\,R = \mathsf{WCM}\,R$. This impression comes from difficulty of the small CM conjecture. Instead, we give another formulation.

We assume that R is a homomorphic image of a CM local ring, and use the *Cohen–Macaulay heart* $\mathcal{H}_{\mathsf{CM}}$ of R introduced in [3]. This is the heart of some compactly generated generated t-structure in the derived category $\mathsf{D}(R)$. This $\mathcal{H}_{\mathsf{CM}}$ is a Grothendieck category with some nice properties. In particular, the Govorov–Lazard type theorem holds for $\mathcal{H}_{\mathsf{CM}}$ in the following sense: Every object in $\mathcal{H}_{\mathsf{CM}}$ is a direct limit of finitely presented objects in $\mathcal{H}_{\mathsf{CM}}$, that is,

$$\varinjlim \operatorname{fp}(\mathcal{H}_{\mathsf{CM}}) = \mathcal{H}_{\mathsf{CM}},$$

where $\operatorname{fp}(\mathcal{H}_{\mathsf{CM}})$ denotes the subcategory of finitely presented objects in $\mathcal{H}_{\mathsf{CM}}$. This fact enables us to show the following result.

Proposition 1. *Let R be a homomorphic image of a CM local ring. Then every weak big CM module is a direct limit of finitely presented objects in $\mathcal{H}_{\mathsf{CM}}$.*

Remark 1. *If R admits a dualizing complex D (such that $\inf\{i \mid H^i(D) \neq 0\} = 0$), there is an equivalence*

$$\mathrm{RHom}_R(-, D) : (\mathsf{mod}\, R)^{\mathrm{op}} \xrightarrow{\sim} \mathrm{fp}(\mathcal{H}_{\mathsf{CM}}),$$

where $\mathrm{fp}(\mathcal{H}_{\mathsf{CM}})$ denotes the subcategory of finitely presented objects in $\mathcal{H}_{\mathsf{CM}}$.

Proposition 1 promises that a similar result to Corollary 1 holds for non-CM case. In fact, we can show the following result without the proposition. Recall that a module M over a local ring $(R, \mathfrak{m})$ is called *derived complete* if the canonical morphism $M \to \mathrm{L}\Lambda^{\mathfrak{m}} M$ is an isomorphism in $\mathsf{D}(R)$, where $\Lambda^{\mathfrak{m}} = \varprojlim_{n \geq 1}(- \otimes_R R/\mathfrak{m}^n)$.

Theorem 2. *Let R be a complete noetherian local ring. Then every pure-injective derived complete big CM module is a direct summand of a direct product of finitely generated R-module. More generally, every derived complete big CM module is a submodule of a direct product of finitely generated R-module.*

Corollary 2. *Let R be a complete noetherian local ring. A big CM R-module is complete if and only if it is derived complete.*

References

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