AFFINE AND UNIRATIONAL UNIQUE FACTORIAL DOMAINS WITH UNMIXED GRADINGS

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This is a joint work with Gene Freudenburg and based on [2]. Throughout the paper, the word "ring" means a commutative ring with unity .

For a ring A and integer $n \geq 1$, $A^{[n]}$ denotes the polynomial ring in n variables over A. If A is an integral domain, $Q(A)$ denotes the quotient field of A. An integral domain A is said to be **rational** (resp. **unirational**) over a field k if $Q(A) \cong Q(k^{[n]})$ (resp. $Q(A) \subseteq Q(k^{[n]})$) for some $n \in \mathbb{N}$.

Let k be a field and $G \cong \mathbb{Z}^n$ for some integer $n \geq 0$ and let A be an integral domain over k with G-grading $A = \bigoplus_{g \in G} A_g$. The grading is **effective** if the weight monoid $M = \{g \in \mathbb{Z}^n \mid A_g \neq 0\}$ generates G as a group, and **unmixed** if for $g, h \in M$, $g + h = 0$ implies $g = h = 0$; see [5], 1.5. If A is finitely generated of dimension n over k and the grading is effective, then A is the coordinate ring of an affine toric variety. These rings have been studied for a long time and can be described combinatorially using cones.

This paper investigates unique factorization domains (UFDs) A which are either finitely generated or unirational over k, and admit an effective unmixed $\mathbb{Z}^{d-1}$-grading with $A_0 = k$, where d is the dimension of A and k is algebraically closed. This means that the variety $V = \operatorname{Spec} A$ admits a torus action of complexity one. Several authors have classified such rings subject to certain additional hypotheses. Mori [6] and Ishida [4] classified them for dimensions $d = 2, 3$, respectively, in case A is finitely generated. Hausen, Herppich and Süss [3] classified them for all dimensions in terms of Cox rings, under the additional assumption that k is algebraically closed of characteristic zero and A is finitely generated. Our main result is the classification of these rings with no additional hypotheses.

To this end, assume that the following data Δ are given.

(Δ.1) An integer $n \geq 2$ and partition $n = n_0 + \cdots + n_r$ where $r, n_i \geq 1$, $0 \leq i \leq r$.

(Δ.2) A sequence $\mathbb{B}_i = (\beta_{i1}, \ldots, \beta_{in_i}) \in (\mathbb{Z}_{\geq 1})^{n_i}$, $0 \leq i \leq r$, where $\gcd(d_i, d_j) = 1$ when $i \neq j$, $d_i = \gcd(\beta_{i1}, \ldots, \beta_{in_i})$.

(Δ.3) A sequence of distinct elements $\lambda_2, \ldots, \lambda_r \in k \setminus \{0\}$.

Any such data set Δ will be called **trinomial data** over k.

Given trinomial data Δ over k, define the ring $k[\Delta]$ as follows. Let $k^{[n]} = k[\mathbb{T}_0, \ldots, \mathbb{T}_r]$ be the polynomial ring in n variables over k, where $\mathbb{T}_i = \{T_{i1}, \ldots, T_{in_i}\}$, $0 \leq i \leq r$, defines a partition of the set of variables T_{ij}. Given i, let $\mathbb{T}_i^{\mathbb{B}_i}$ denote the monomial $T_{i1}^{\beta_{i1}} \cdots T_{in_i}^{\beta_{in_i}}$. We then define:

$$k[\Delta] = k[\mathbb{T}_0, \ldots, \mathbb{T}_r]/(\mathbb{T}_0^{\mathbb{B}_0} + \lambda_i \mathbb{T}_1^{\mathbb{B}_1} + \mathbb{T}_i^{\mathbb{B}_i})_{2 \leq i \leq r}$$

Observe the following.

(1) If $r = 1$, then λ_i is the empty sequence and $k[\Delta] = k^{[n]}$.
(2) If K is an extension field of k, then $K \otimes_k k[\Delta] = K[\Delta]$.

Recently, we showed the following.

Theorem 1.1. ([1], Theorem 5.1) *Let k be any field. Given trinomial data Δ over k, $k[\Delta]$ is an finitely generated rational UFD of dimension $n - r + 1 \geq 2$ over k.*

The following is the main result in this paper. Our main result builds on Theorem 1.1.

Theorem 1.2. *Suppose that k is an algebraically closed field and A is an integral domain of finite transcendence degree $d \geq 2$ over k. The following conditions (1), (2) and (3) for A are equivalent.*

(1) *$A \cong_k k[\Delta]^{[m]}$ for some trinomial data Δ over k and some $m \in \mathbb{N}$.*
(2) *A is a unirational UFD which admits an effective unmixed $\mathbb{Z}^{d-1}$-grading with $A_0 = k$.*
(3) *A is a finitely generated UFD which admits an effective unmixed $\mathbb{Z}^{d-1}$-grading with $A_0 = k$.*

When k is algebraically closed, the following corollary shows that the only smooth varieties among those of the form $V = \mathrm{Spec}(k[\Delta])$ for trinomial data Δ are the affine spaces.

Corollary 1.3. *Assume that k is algebraically closed. Given trinomial data Δ over k, if $V = \mathrm{Spec}(k[\Delta])$ is smooth then $V \cong_k \mathbb{A}^n_k$. That is, $k[\Delta] \cong_k k^{[n]}$.*

The following corollary shows that the Zariski cancellation problem holds for an algebra A which admits an effective unmixed $\mathbb{Z}^{d-1}$-grading with $A_0 = k$.

Corollary 1.4. *Let $d \geq 2$. Suppose that k is an algebraically closed field and A is an integral domain satisfying $A^{[1]} \cong_k k^{[d+1]}$. If A admits an effective unmixed $\mathbb{Z}^{d-1}$-grading with $A_0 = k$, then $A \cong_k k^{[d]}$.*

Proof. Suppose that $A^{[1]} \cong_k k^{[d+1]}$. Then A is a unirational UFD of transcendence degree d over k. By Theorem 1.2, $A \cong_k k[\Delta]^{[m]}$ for some trinomial data Δ over k and some $m \in \mathbb{N}$. Since $k[\Delta]^{[m+1]} \cong_k A^{[1]} \cong_k k^{[d+1]}$, $V = \mathrm{Spec}(k[\Delta])$ is smooth. It follows from Corollary 1.3 that $k[\Delta] \cong_k k^{[d-m]}$ and hence $A \cong_k k^{[d]}$. □

References

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