COMPARING GENERALIZATIONS OF GORENSTEINNESS IN SEMI-STANDARD GRADED RINGS

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Let $R = \bigoplus_{i \in \mathbb{N}} R_i$ be a graded Noetherian $\Bbbk$-algebra over a field $R_0 = \Bbbk$. If $R = \Bbbk[R_1]$, that is, R is generated by R_1 as a $\Bbbk$-algebra, then we say R is *standard graded*. If R is finitely generated as a $\Bbbk[R_1]$-module, then we say R is *semi-standard graded*. The *Ehrhart rings* of lattice polytopes and the face rings of simplicial posets (see [9]) are typical classes of semi-standard graded rings. From perspective of combinatorial commutative algebra, the concept of semi-standard graded rings naturally arises in this context.

If R is a semi-standard graded ring of dimension d, its Hilbert series is of the form

$$\sum_{i \in \mathbb{N}} (\dim_{\Bbbk} R_i) t^i = \frac{h_0 + h_1 t + \cdots + h_s t^s}{(1-t)^d}$$

for some integers $h_0, h_1, \cdots, h_s$ with $\sum_{i=0}^{s} h_i \neq 0$ and $h_s \neq 0$. We call the integer sequence $(h_0, h_1, \cdots, h_s)$ the *h-vector* of R. We call s the *socle degree* of R and denote it as $s(R)$.

With the development of non-Gorenstein Cohen-Macaulay analysis, various generalized properties of Gorenstein rings have been defined. Notable examples include nearly Gorenstein, almost Gorenstein, and level property. Comparisons of these properties have been done in [2, 4, 6, 8]. In this talk, we apply the techniques developed for standard graded rings, as seen in [3, 6], to the case of semi-standard graded rings. We establish the theory for semi-standard graded rings and extend several well-known results about standard graded rings to the case of semi-standard graded rings. For instance, We extend [6, Theorem 4.4] further and demonstrate the following statement regarding trace ideals of semi-standard graded rings. This is applicable even in the non-Cohen-Macaulay case.

Theorem 1. *Let $R = \Bbbk[S]$ be a semi-standard graded affine semigroup ring. Let I be a non-principle ideal of R and let $b = \min\{i : I_i \neq 0\}$. If depth$R \geqq 2$ and*

$$(\mathbf{x}^{\mathbf{e}} : \mathbf{e} \in E_S)R \subseteq tr(I),$$

then we have $\dim_{\Bbbk} I_b \geqq 2$.

Corollary 2. *Let $R = \Bbbk[S]$ be a semi-standard graded Cohen-Macaulay affine semigroup ring. If R is not Gorenstein nearly Gorenstein, then $h_{s(R)} \geqq 2$. In particular, if R is nearly Gorenstein with Cohen-Macaulay type 2, then R is level.*

At the same time, we investigate specific properties that do not hold in standard graded rings but hold for semi-standard graded rings. For instance, there are intriguing differences if we consider nearly Gorenstein affine semigroup rings with projective dimension 2. In the case of standard graded affine semigroup rings, there exist non-Gorenstein yet nearly Gorenstein rings with projective dimension 2, and their characterization is provided for the case of projective monomial curves (see [7, Theorem A]). However, in the case of non-standard semi-standard graded rings, it turns out that there is no non-Gorenstein nearly Gorenstein ring with projective dimension 2.

Theorem 3. *Let R be a non-standard semi-standard graded Cohen-Macaulay affine semigroup ring with projective dimension 2. Then the following conditions are equivalent:*

(1) *R is nearly Gorenstein;*
(2) *R is Gorenstein.*

The following theorem regarding the necessary and sufficient condition to be level and almost Gorenstein was also known in the case of standard graded rings, as shown in [1]. We give a new proof by using Stanley's inequalities, and extend their results to the case of semi-standard graded rings.

Theorem 4. *Let R be a Cohen–Macaulay semi-standard graded ring with $\dim R > 0$. Suppose that R is not Gorenstein. Then the following conditions are equivalent:*

(1) *R is almost Gorenstein and level;*
(2) *R is generically Gorenstein and $s(R) = 1$.*

Moreover, in standard graded affine semigroup rings, it is known that there is no instance that it is non-level almost Gorenstein and nearly Gorenstein. However, in the case of semi-standard graded affine semigroup rings, such a special family is known to exist when the socle degree is 2. For the case of dimension 2, this family can be characterized as follows. The proof of this assertion relies significantly on the proof presented in [5, Theorem 3.5].

Theorem 5. *Let $R = \mathbb{k}[S]$ be a Cohen–Macaulay semi-standard graded affine semigroup ring with $\dim R = s(R) = 2$. Then the following conditions are equivalent:*

(1) *R is non-level and almost Gorenstein;*
(2) *$S \cong \langle \{(2i, 2n-2i) : 0 \leqq i \leqq n\} \bigcup \{(2j+2k-1, 4n-2j-2k+1) : 0 \leqq j \leqq n-1\} \rangle$ for some $n \geqq 2$ and $1 \leqq k \leqq n+1$.*

Moreover, if this is the case, then R is always nearly Gorenstein and $h(R) = (1, n-1, n)$.

Corollary 6. *Let R be a non-standard semi-standard graded Cohen-Macaulay affine semigroup ring with $\dim R = s(R) = 2$. If R is almost Gorenstein, then it is nearly Gorenstein.*

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