Extended version of abstract for my talk "The Koszul Property for Algebras Associated to Matroids"

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Let M be a matroid on base set $E = \{e_1, \dots, e_n\}$ with associated lattice of flats $\mathcal{L}(M)$. Matroids are generalization of vector configurations or hyperplane arrangements that encode independence combinatorially. Several constructions of algebras associated to hyperplane arrangements can be defined purely in terms of the lattice of flats and thus extend to all matroids. In this talk, I will survey the Koszul property for the following four $\mathbb{Q}$-algebras:

1. The **Orlik-Solomon algebra** of M, denoted $A(M)$ is

$$A(M) := \frac{\bigwedge_{\mathbb{Q}}\langle y_1, \dots, y_n\rangle}{(\partial(y_C) \mid C \text{ is a circuit of } M)},$$

   where $C = \{y_{i_1}, \dots, y_{i_t}\}$, $y_C = \prod_{j=1}^{t} y_{i_j}$ and $\partial(y_C) = \sum_{j=1}^{t} y_{i_1} \cdots \widehat{y_{i_j}} \cdots y_{i_t}$.

2. The **Graded Möbius algebra** of M, denoted $H(M)$, is the commutative $\mathbb{Q}$-algebra

$$HM := \bigoplus_{F \in \mathcal{L}(M)} \mathbb{Q} y_F$$

   having the elements y_F for each flat F of M as a $\mathbb{Q}$-basis, with multiplication defined by

$$y_F y_G = \begin{cases} y_{F \vee G}, & \text{if } \operatorname{rank}(F \vee G) = \operatorname{rank} F + \operatorname{rank} G \\ 0, & \text{otherwise.} \end{cases}$$

3. The **Chow ring** of M, denoted $\underline{\mathrm{CH}}(M)$, is

$$\underline{\mathrm{CH}}(M) := \underline{S}_M / (\underline{I}_M + \underline{J}_M),$$

   where

$$\underline{S}_M = \mathbb{Q}[x_F \mid F \text{ is a nonempty flat of } M],$$

   and $\underline{I}_M$ and $\underline{J}_M$ are the ideals:

$$\underline{I}_M = (\textstyle\sum_{i \in F} x_F \mid i \in E)$$

$$\underline{J}_M = (x_F x_G \mid F, G \text{ are incomparable, nonempty flats of } M)\,.$$

4. The **Augmented Chow ring** of M, denoted $\mathrm{CH}(M)$, is

$$\mathrm{CH}(M) := S_M/(I_M + J_M),$$

where

$$S_M = \mathbb{Q}[y_i, x_F \mid i \in E, F \in \mathcal{L}(M) \smallsetminus \{E\}],$$

and I_M and J_M are the ideals

$$I_M = (y_i - \textstyle\sum_{i \notin F} x_F \mid i \in E),$$

$$J_M = (x_F x_G \mid F, G \text{ incomparable }) + (y_i x_F \mid i \in E, i \notin F).$$

In the case of a representable matroid associated to a complex hyperplane arrangement $\mathcal{H} \subseteq \mathbb{C}^d$, $A(M) \otimes_{\mathbb{Q}} \mathbb{C}$ is isomorphic to the de Rham cohomology ring of the complement $\mathbb{C}^d \smallsetminus \bigcup_{H \in \mathcal{H}} H$. Björner-Ziegler and Peeva showed that $A(M)$ has a quadratic Gröbner basis if and only if $\mathcal{L}(M)$ is supersolvable. Thus if $\mathcal{L}(M)$ is supersolvable, then $A(M)$ is Koszul while the converse is still open. For a graphic matroid $M(G)$, $\mathcal{L}(M(G))$ is supersolvable if and only if $A(M(G))$ is Koszul if and only if G is chordal by work of Stanley.

The other three algebras are newer and all play a roll in the recent resolution of the Dowling-Wilson Top Heavy Conjecture by Braden et. al.. Matroeni, Peeva, and the author showed that $H(M)$ has a quadratic Gröbner basis if and only if M satisfies a condition called T-chordality. For graphic matroids, we show that quadracity $H(M(G))$ is equivalent to G being chordal, however there are chordal graphics for which $H(M(G))$ is not Koszul. We show that if $H(M(G))$ is Koszul, then G is strongly chordal, and we conjecture that the converse holds, but this is still open. We prove the converse for several classes of strongly chordal graphs including proper interval graphs and threshold graphs.

The Chow ring and Augmented Chow rings are known to satisfy the Kähler package; namely, they have Poincaré duality and satisfy versions of the with versions of the hard Lefschetz and Hodge-Riemann conditions. In particular, they are artinian, quadratic Gorenstein rings. Dotsenko conjectured that the former was Koszul. In joint work with Mastroeni, we proved Dotsenko's conjecture for all matroids using a filtration argument. We also show all augmented Chow rings of matroids are Koszul.

References

[1] T. Braden, J. Huh, J. Matherne, N. Proudfoot, and B. Wong. Singular hodge theory for combinatorial geometries, 2020. arXiv.CO 2010.06088

[2] M. Mastroeni and J. McCullough. Chow rings of matroids are Koszul. *Math. Ann.* 387, 1819-1851 (2023).

[3] M. Mastroeni, J. McCullough, and I. Peeva. Koszul graded Möbius Algebras preprint. (2023)