NON-COMMUTATIVE CREPANT RESOLUTIONS OF A SPECIAL FAMILY OF STABLE SET RINGS

KOJI MATSUSHITA

This talk is based on [7]. Let R be a Gorenstein normal domain. For a reflexive R-module $M \neq 0$, we set $E = \mathrm{End}_R(M)$ and denote the global dimension of E by $\mathrm{gldim}\, E$. We call E a *non-commutative crepant resolution* (*NCCR*, for short) of R if E is a maximal Cohen-Macauly (MCM, for short) R-module with $\mathrm{gldim}\, E < \infty$. Moreover, we say that an NCCR E is *splitting* if M is a finite direct sum of rank one reflexive R-modules. A splitting NCCR is also called "toric NCCR" when R is a toric ring. The existence of NCCRs of toric rings is the one of the most well-studied problems.

Recently, conic divisorial ideals, which are a certain class of divisorial ideals (rank one reflexive modules) defined on toric rings, and their applications are well studied (see e.g., [2, 3], and so on). Let $R = \Bbbk[C \cap \mathbb{Z}^d]$ be the toric ring of a strongly convex rational polyhedral cone $C \subset \mathbb{R}^d$, where $\Bbbk$ is an algebraically closed field of characteristic 0. It is known that up to isomorphism the conic divisorial ideals of R are exactly the direct summands of $R^{1/k}$ for $k \gg 0$ ([3, Proposition 3.6], [9, Proposition 3.2.3]), where $R^{1/k} = \Bbbk[C \cap (1/k\mathbb{Z})^d]$ is regarded as an R-module. In addition, conic divisorial ideals of R are MCM R-modules. Since conic divisorial ideals of a toric ring R are rank one reflexive MCM R-modules, the endomorphism ring E of the finite direct sum of some of them is a toric NCCR if E is an MCM R-module and $\mathrm{gldim}\, E < \infty$.

It is known that the following toric rings have toric NCCRs:

- Gorenstein toric rings whose divisor class groups are $\mathbb{Z}$ ([12]);
- Gorenstein Hibi rings whose divisor class groups are $\mathbb{Z}^2$ ([8]);
- 3-dimensional Gorenstein toric rings ([1, 6, 11]);
- Segre products of polynomial rings ([5]);
- Gorenstein edge rings of complete multipartite graphs ([4]).

On the other hand, there is an example of a Gorenstein toric ring which have no toric NCCRs ([10, Example 9.1]). Therefore, it is interesting to ask when a Gorenstein toric ring has a toric NCCR.

In this talk, we give an NCCR of a special family of stable set rings, which are toric rings arising from graphs.

Let G be a simple graph on the vertex set $V(G) = [d] := \{1, \dots, d\}$ with the edge set $E(G)$. We say that $S \subset V(G)$ is a *stable set* (resp. a *clique*) if $\{v, w\} \notin E(G)$ (resp. $\{v, w\} \in E(G)$) for any distinct vertices $v, w \in S$. Note that the empty set and each singleton are regarded as stable sets.

We define the *stable set ring* of G over $\Bbbk$ by setting

$$\Bbbk[\mathrm{Stab}_G] = \Bbbk[\ (\textstyle\prod_{i \in S} t_i)t_0 : S \text{ is a stable set of } G].$$

The stable set ring of G can be described as the toric ring arising from a rational polyhedral cone if G is perfect.

For an integer $n \geq 3$ and positive integers $r_1, \dots, r_n$, let $G_{r_1,\dots,r_n}$ be the graph on the vertex set $V(G_{r_1,\dots,r_n}) = [2d]$ with the edge set $E(G_{r_1,\dots,r_n}) = \bigcup_{i=0}^{n} \big\{\{v, u\} : v, u \in Q_i\big\}$, where $d = \sum_{k=1}^{n} r_k$, $Q_0 = \{d+1, \dots, 2d\}$ and for $i \in [n]$, we let

$$Q_i^+ = \Big\{ \sum_{k=1}^{i-1} r_k + 1, \dots, \sum_{k=1}^{i} r_k \Big\},\ Q_i^- = \Big\{ d + \sum_{k=1}^{i-1} r_k + 1, \dots, d + \sum_{k=1}^{i} r_k \Big\} \text{ and } Q_i = Q_i^+ \cup (Q_0 \setminus Q_i^-).$$

Note that $Q_i^+ = Q_i \setminus Q_0$ and $Q_i^- = Q_0 \setminus Q_i$. Moreover, we set

$$\mathcal{C}(G_{r_1,\dots,r_n}) = \{(z_1, \cdots, z_n) \in \mathbb{R}^n : -r_i \le z_i \le r_i \text{ for } i \in [n]\}.$$

The graph $G_{r_1,\dots,r_n}$ has the following properties.

Proposition 1. (i) *The maximal cliques of $G_{r_1,\dots,r_n}$ are precisely $Q_0, Q_1, \dots, Q_n$.*

(ii) *A subset $S \subset V(G_{r_1,\dots,r_n})$ is a maximal stable set of $G_{r_1,\dots,r_n}$ if and only if $S = \{v_i, v_i'\}$ or $\{v_1, \dots, v_n\}$ for some $i \in [n]$, $v_i \in Q_i^+$ and $v_i' \in Q_i^-$.*

(iii) *The graph $G_{r_1,\dots,r_n}$ is chordal, and hence it is perfect.*

(iv) *The stable set ring $\Bbbk[\mathrm{Stab}_{G_{r_1,\dots,r_n}}]$ is Gorenstein, and its divisor class group is isomorphic to $\mathbb{Z}^n$.*

(v) *Each point in $\mathcal{C}(G_{r_1,\dots,r_n}) \cap \mathbb{Z}^n$ one-to-one corresponds to the conic divisorial ideal of $\Bbbk[\mathrm{Stab}_{G_{r_1,\dots,r_n}}]$*

Theorem 2. *Let $R = \Bbbk[\mathrm{Stab}_{G_{r_1,\dots,r_n}}]$ and*

$$\mathcal{L} = \{(z_1, \cdots, z_n) \in \mathbb{Z}^n : 0 \le z_i \le r_i \text{ for } i \in [n]\} \subset \mathcal{C}(G_{r_1,\dots,r_n}) \cap \mathbb{Z}^n.$$

Moreover, let $M_{\mathcal{L}}$ be the direct sum of the conic divisorial ideals coressponding to the elements of $\mathcal{L}$. Then, $E = \mathrm{End}_R(M_{\mathcal{L}})$ is an NCCR of R. In particular, R has a toric NCCR.

References

[1] N. Broomhead, Dimer model and Calabi-Yau algebras, Mem. Amer. Math. Soc., **215** no. 1011, (2012).

[2] W. Bruns, Conic divisor classes over a normal monoid algebra, *Commutative algebra and algebraic geometry, Contemp. Math.*, **390**, Amer. Math. Soc., (2005), 63-71.

[3] W. Bruns and J. Gubeladze, Divisorial linear algebra of normal semigroup rings, *Algebra and Represent. Theory*, **6** (2003), 139-168.

[4] A. Higashitani and K. Matsushita, Conic divisorial ideals and non-commutative crepant resolutions of edge rings of complete multipartite graphs, *J. Algebra* **594** (2022), 685–711.

[5] A. Higashitani and Y. Nakajima, Conic divisorial ideals of Hibi rings and their applications to non-commutative crepant resolutions, *Selecta Math.* **25** (2019), 25pp.

[6] A. Ishii and K. Ueda, Dimer models and the special McKay correspondence, *Geom. Topol.* **19** (2015) 3405–3466.

[7] K. Matsushita, Conic divisorial ideals of toric rings and applications to Hibi rings and stable set rings, arXiv:2210.02031.

[8] Y. Nakajima, Non-commutative crepant resolutions of Hibi rings with small class group, *J. Pure Appl. Algebra*, **223** (2019), no. 8, 3461–3484.

[9] K. E. Smith and M. Van den Bergh, Simplicity of rings of differential operators in prime characteristic, *Proc. London Math. Soc.* (3) **75** (1997), no. 1, 32–62.

[10] Š. Špenko and M. Van den Bergh, Non-commutative resolutions of quotient singularities for reductive groups, *Invent. Math.* **210** (2017), no. 1, 3–67.

[11] Š. Špenko and M. Van den Bergh, Non-commutative crepant resolutions for some toric singularities I, *Int. Math. Res. Not. IMRN* (2020), no. 21, 8120-8138.

[12] M. Van den Bergh, Non-Commutative Crepant Resolutions, *The Legacy of Niels Henrik Abel*, Springer-Verlag, Berlin, (2004), 749–770.

Department of Pure and Applied Mathematics, Graduate School of Information Science and Technology, Osaka University, Suita, Osaka 565-0871, Japan

Email address: k-matsushita@ist.osaka-u.ac.jp