Limit Frobenius complexity and toric face rings

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Let R be a commutative ring with prime characteristic p and M an R-module. We denote by ${}^eM$ the R-module whose additive group structure is that of M and the action of R is defined by e times iterated Frobenius map: $r \cdot m = r^{p^e}m$, where the right hand side is the original action of R on M. We set $\mathcal{F}^e(M) := \operatorname{Hom}_R(M, {}^eM)$. It is easily verified that if $\varphi \in \mathcal{F}^e(M)$ and $\psi \in \mathcal{F}^{e'}(M)$, then $\psi \circ \varphi \in \mathcal{F}^{e+e'}(M)$. Therefore, the module $\bigoplus_{e\geq 0} \mathcal{F}^e(M)$ has a structure of non-commutative R-algebra structure. We denote this ring by $\mathcal{F}(M)$ and call the ring of Frobenius operators on M.

Let $A = \bigoplus_{n\geq 0} A_n$ be an $\mathbb{N}$-graded not necessarily commutative ring. For $e \geq 0$, we denote by $G_e(A)$ the subring of A generated by homogeneous elements of A with degree at most e. For $e \geq 1$, we denote by $c_e(A)$ the minimal number of elements which generate $A_e/G_{e-1}(A)_e$ as a two-sided A_0-module. If $c_e(A)$ is finite for any e, we say that A is degree-wise finitely generated.

Suppose that A is degree-wise finitely generated. We define the complexity $\operatorname{cx}(A)$ of A by

$$\operatorname{cx}(A) := \inf\{n \in \mathbb{R}_{>0} \mid c_e(A) = O(n^e)\ (e \to \infty)\}$$

if $\{n \in \mathbb{R}_{>0} \mid c_e(A) = O(n^e)\ (e \to \infty)\} \neq \emptyset$ and $\operatorname{cx}(A) := \infty$ otherwise, where O is the Landau symbol.

Let $R = \bigoplus_{n\geq 0} R_n$ be an $\mathbb{N}$-graded Noetherian commutative ring with $R_0 = \mathbb{K}$, a field of prime characteristic p. Set $\mathfrak{m} = \bigoplus_{n>0} R_n$ and let E be the injective hull of $R/\mathfrak{m}$ as an R-module. Enescu and Yau [EY1] called $\log_p \operatorname{cx}(\mathcal{F}(E))$ the Frobenius complexity of R. The Frobenius complexity of R is denoted by $\operatorname{cx}_F(R)$. They showed [EY2] that if R is the Segre product of polynomial rings with m and n variables over $\mathbb{K}$, where $m > n \geq 2$, then $\operatorname{cx}_F(R) \to m-1$ as $p \to \infty$. Page [Pag] generalized this result and showed that if R is a non-Gorenstein anticanonical level Hibi ring on a distributive lattice H over $\mathbb{K}$ and P is the set of join-irreducible elements of H, then $\operatorname{cx}_F(R) \to \#P_{\mathrm{nonmin}}$ as $p \to \infty$, where $P_{\mathrm{nonmin}} = \{z \in P \mid z \text{ is not in any maximal chain of minimal length}\}$.

In these cases, R is normal and $\operatorname{cx}_F(R) \to \dim \bigoplus_{n\geq 0} \omega^{(-n)}/\mathfrak{m}\omega^{(-n)} - 1$ as $p \to \infty$, where $\omega^{(-n)} = R :_{Q(R)} \omega^n$ for $n \geq 0$. Note that it always holds that $\operatorname{cx}_F(R) \leq \dim \bigoplus_{n\geq 0} \omega^{(-n)}/\mathfrak{m}\omega^{(-n)} - 1$ for any p and the inequalities are strict even in the above cases for a fixed p.

Page made a question that given a Hibi ring R (or more generally, any toric ring) over a field of characteristic p, is it always true that $\lim_{p\to\infty} \operatorname{cx}_F(R) = \dim \bigoplus_{n\geq 0} \omega^{(-n)}/\mathfrak{m}\omega^{(-n)} - 1$? The present author showed that the Hibi ring case of Page's question is affirmative [Miy].

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In this talk, we first show a counterexample of the general case of Page's question. Further, since the fiber cones $\bigoplus_{n\geq 0}\omega^{(-n)}/\mathfrak{m}\omega^{(-n)}$ and $\bigoplus_{n\geq 0}\omega^{(n)}/\mathfrak{m}\omega^{(n)}$ of the anticanonical and the canonical symbolic Rees rings are toric face rings, we exhibit several examples of these rings.

References

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