The divisor class group of a complete local log-regular ring

Shinnosuke Ishiro (Tokyo Institute of Science)

This talk is based on [5]. The paper [5] combines the content of preprint [4] with the content of this talk.

The studies on singularities in mixed characteristic have been rapidly developed in recent years (for example, see [7], [2]). The main tools of these studies are big Cohen–Macaulay algebras and perfectoid rings, which are classes of non-Noetherian rings. Hence we can not usually compute various invariants appearing singularities in mixed characteristic. To find typical examples that we can compute their invariants is a very important problem of commutative ring theory in mixed characteristic. To give one solution to this problem, we have studied the ring-theoretic properties of local log-regular rings. The class of local log-regular rings is defined by Kazuya Kato in [6] to develop the theory of toric varieties without bases. The advantage of this class is to have properties similar to those of toric rings though it can be defined in characteristic-free.

In this abstract, a *monoid* means a commutative semigroup with unity. We denote its binary operation by $+$ and the unit of a monoid by 0. For a monoid $\mathcal{Q}$, we define $\mathcal{Q}^{\mathrm{gp}}$ as the group whose elements form $x - y$ where $x, y \in \mathcal{Q}$. We also define $\mathcal{Q}^*$ as the group of units of $\mathcal{Q}$. Here we introduce the log-structure of commutative rings.

Definition 1. *Let R be a ring, let $\mathcal{Q}$ be a monoid, and let $\alpha : \mathcal{Q} \to R$ be a monoid homomorphism where we regard R as a multiplicative monoid. Then we say that the triple $(R, \mathcal{Q}, \alpha)$ is a log ring*[1]*. Moreover, a log ring $(R, \mathcal{Q}, \alpha)$ is called local if R is local and $\alpha^{-1}(R^\times) = \mathcal{Q}^*$.*

Definition 2. *Let $(R, \mathcal{Q}, \alpha)$ be a local log ring, where R is Noetherian and $\overline{\mathcal{Q}} := \mathcal{Q}/\mathcal{Q}^*$ is fine and saturated. Let I_α be the ideal of R generated by $\alpha(\mathcal{Q}^+)$. Then $(R, \mathcal{Q}, \alpha)$ is called a local log-regular ring if the following conditions are satisfied:*

1. *R/I_α is a regular local ring.*
2. *The equality $\dim R = \dim(R/I_\alpha) + \dim \mathcal{Q}$ holds.*

One of the main tools for exploring properties of local log-regular rings is the structure theorem which is an analogue of Cohen's structure theorem. In addition, local log-regular rings have good ring-theoretic properties such as Cohen–Macaulayness, normality, and so on.

Here, we state the main result of this talk. Our main result is that the divisor class group of a local log-regular ring is isomorphic to that of the associated monoid. We omit the definition of the divisor class group of an integral monoid in this abstract because it can be defined in the same manner as that of an integral domain.

[1] We sometimes denote a log ring by $\alpha : R \to \mathcal{Q}$ to be aware of the structure morphism.

Theorem 3 (cf. [3, Corollary 12.6.43] or [4])**.** *Let* $(R, \mathcal{Q}, \alpha)$ *be a complete local log-regular ring. Then the monoid homomorphism* α *induces the group isomorphism*

$$\mathrm{Cl}(\mathcal{Q}) \xrightarrow{\cong} \mathrm{Cl}(R)\ ;\ \mathrm{cl}(\mathfrak{p}) \mapsto \mathrm{cl}(\mathfrak{p}R).$$

Theorem 3 is already proved by Gabber and Ramero in [3], but they proved it by a method in algebraic geometry, not commutative ring theory. In contrast, we prove it using only commutative ring theory. Combining the isomorphism $\mathrm{Cl}(k[\mathcal{Q}]) \cong \mathrm{Cl}(\mathcal{Q})$ due to Chouinard [1] with Theorem 3, we obtain the following corollary.

Corollary 4. *Let* $(R, \mathcal{Q}, \alpha)$ *be a local log-regular ring, and let* $\widehat{R}$ *be the completion of* R *with respect to the maximal ideal. Then* $\mathrm{Cl}(\widehat{R})$ *is isomorphic to* $\mathrm{Cl}(k[\mathcal{Q}])$ *where* k *is any field, namely,* $\mathrm{Cl}(\widehat{R})$ *is finitely generated.*

In our talk, we begin to introduce basic properties of local log-regular rings which have been proven, and give examples. After that, we give the sketch of our proof of Theorem 3.

References

[1] Chouinard II, Leo G, *Krull semigroups and divisor class groups*, Canadian Journal of Mathematics 33.6 (1981): 1459–1468.

[2] Hanlin Cai, Seungsu Lee, Linquan Ma, Karl Schwede, and Kevin Tucker, *Perfectoid signature, perfectoid Hilbert–Kunz multiplicity, and an application to local fundamental groups*, arXiv preprint arXiv:2209.04046, 2022.

[3] O. Gabber and L. Ramero, *Almost rings and perfectoid spaces*, `https://pro.univ-lille.fr/fileadmin/user_upload/pages_pros/lorenzo_ramero/hodge.pdf`, Release 8, 2023.

[4] Shinnosuke Ishiro, *The canonical module of a local log-regular ring*, arXiv preprint arXiv:2209.04828, 2022

[5] Shinnosuke Ishiro, *Some ring-theoretic properties of local log-regular rings*, in preparation, 2023.

[6] Kazuya Kato, *Toric singularities*, American Journal of Mathematics, 116(5):1073–1099, 1994.

[7] Linquan Ma and Karl Schwede, *Singularities in mixed characteristic via perfectoid big Cohen–Macaulay algebras*, Duke Mathematical Journal, 170(13):2815–2890, 2021.