DIFFERENCE OF HILBERT SERIES OF HOMOGENEOUS AFFINE SEMIGROUP RING AND ITS NORMALIZATION

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Let $Q \subset \mathbb{Z}^d_{\geq 0}$ be an *affine semigroup*, which is a finitely generated sub-semigroup of $\mathbb{Z}^d_{\geq 0}$. We say that Q is *homogeneous* if the minimal generating set of Q lies on the same hyperplane not containing the origin. The *normalization* of Q is the semigroup of the form $\overline{Q} = \mathbb{Z}Q \cap \mathbb{R}_{\geq 0}Q$, where $\mathbb{Z}Q$ (resp. $\mathbb{R}_{\geq 0}Q$) denotes the free abelian group (resp. the polyhedral cone) generated by Q. We say that Q is *normal* if $\overline{Q} = Q$. Let $\Bbbk$ be a field. We denote by $\Bbbk[Q]$ the associated semigroup ring of Q. Note that $\Bbbk[\overline{Q}] = \overline{\Bbbk[Q]}$ and $\Bbbk[\overline{Q}]$ is a finitely generated $\Bbbk[Q]$-module. For the introduction to the theory of affine semigroup rings, see, e.g., [1, Section 6].

Given a $\mathbb{Z}$-graded $\Bbbk$-algebra $R = \bigoplus_{i\in\mathbb{Z}} R_i$, let $M = \bigoplus_{i\in\mathbb{Z}} M_i$ be a finitely generated $\mathbb{Z}$-graded R-module with $\dim_\Bbbk M_i < \infty$ for each i. Let $\mathrm{Hilb}(M,t) = \sum_{i\in\mathbb{Z}} \dim_\Bbbk M_i t^i$ denote the Hilbert series of M. If R is homogeneous, i.e., R is generated by R_1 and $R_0 = \Bbbk$, then we see that $\mathrm{Hilb}(M,t)$ is of the form $\mathrm{Hilb}(M,t) = \dfrac{h_M(t)}{(1-t)^{\dim M}}$, where $h_M(t) \in \mathbb{Z}[t^{\pm}]$ with $h_M(1) \neq 0$ such that the least degree of $h_M(t)$ is equal to the least index i with $M_i \neq 0$. We call the (Laurent) polynomial $h_M(t)$ appearing in $\mathrm{Hilb}(M,t)$ the *h-polynomial* of M.

If R is $\Bbbk[Q]$ for some homogeneous affine semigroup Q, then we use the notation $\mathrm{Hilb}(Q,t)$ and $h_Q(t)$ instead of $\mathrm{Hilb}(\Bbbk[Q],t)$ and $h_{\Bbbk[Q]}(t)$, respectively.

The following is the first main theorem of this talk.

Theorem 1. *Let Q be a homogeneous affine semigroup and assume that $\Bbbk[Q]$ satisfies Serre's condition (S_2). Then $\deg(h_Q(t)) \geq \deg(h_{\overline{Q}}(t))$.*

Here, $\deg(f(t))$ denotes the degree of the polynomial $f(t)$. A proof of Theorem 1 relies on the structure of holes of Q, i.e., $\overline{Q} \setminus Q$. It is known by [2, Theorem 5.2] that $\Bbbk[Q]$ satisfies Serre's condition (S_2) if and only if $\overline{Q} \setminus Q$ consists of faces of Q of dimension $d-1$.

The following second main theorem shows the existence of counterexamples of Theorem 1 if we drop the assumption (S_2).

Theorem 2. *For any positive integer m, there exists a homogeneous affine semigroup Q such that $\deg(h_{\overline{Q}}(t)) - \deg(h_Q(t)) = m$.*

An example of Q satisfying $\deg(h_{\overline{Q}}(t)) - \deg(h_Q(t)) = m$ is constructed as the join of the edge ring of a certain graph.

References

[1] W. Bruns and J. Herzog, "Cohen-Macaulay rings, revised edition", Cambridge University Press, 1998.
[2] L. Katthän, Non-normal affine monoid algebras, *Manuscr. Math.* **146**, (2015), 223–233.

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