AN EXAMPLE TO A QUESTION OF RATLIFF ON ASYMPTOTIC PRIME DIVISORS

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Let R be a Noetherian domain and let I be an ideal in R. In 1976, Ratliff considered the following statement on the set $\mathrm{Ass}_R(R/I^n)$ of primes associated to powers of I:

If $P \in \mathrm{Ass}_R(R/I^k)$ for some $k \geq 1$, then $P \in \mathrm{Ass}_R(R/I^n)$ for all large n.

Ratliff gave in [Rat76] a number of positive answers including results on the set $\mathrm{Ass}_R(R/\overline{I^n})$ of primes associated to the integral closure of powers of I. In 1979, Brodmann proved that the statement is true if k is large enough, but it is not true in general. More precisely, Brodmann showed that for any integer $m \geq 2$, there exist examples of an affine k-domain R of $\dim R = 2$ and ideals $I \subset P$ in R such that $P \in \mathrm{Ass}_R(R/I^n)$ if and only if $n < m$.

This examples show that the sets $\mathrm{Ass}_R(R/I^n)$ do not necessarily increase in initial part, unlike the sets $\mathrm{Ass}_R(R/\overline{I^n})$ are monotonically increasing and eventually stable. Then it is natural to ask about the initial behavior of the sets $\mathrm{Ass}_R(R/I^n)$. In [Rat83], Ratliff raised the following interesting question:

Question 1 (Ratliff)**.** *Given a finite set S of positive integers, do there exist a Noetherian ring R and ideals $I \subset P$ in R such that $P \in \mathrm{Ass}_R(R/I^n)$ if and only if $n \in S$?*

Note that Brodmann's examples gave a positive answer to this question in a case where $S = \{1, 2, \ldots m-1\}$ is a consecutive integers starting from 1 to $m-1$.

Recently, Hà-Ngyuen-Trung-Trung gave in [HNTT21] a positive answer to this question. In fact, they settled a conjecture of Herzog-Hibi in [HeHi05] on the depth function of powers of ideals, and showed that a positive answer to the Ratliff's question follows as a direct consequence. We point out that they gave how to construct such ideals explicitly and the ideals can be a monomial ideal in a polynomial ring whose number of variables also depends on a given set S in the question.

In this talk, by improving and generalizing the Brodmann's classical example, we will give a new example to the question of Ratliff in a special case where S is any consecutive positive integers. Our example shows the following:

Theorem 2. *For any consecutive integers $S = \{d, d+1, \ldots, d'\}$ starting from a positive integer d to $d' \geq d$, there exist a Noetherian domain R of dimension $d+1$ and ideals $I \subset P$ in R such that $P \in \mathrm{Ass}_R(R/I^n)$ if and only if $n \in S$.*

We point out that the monomial ideals constructed in [HNTT21] for the set S in Theorem 2 uses at least 5 variables. On the other hands, if we take $d = 2$ (resp. $d = 3$) in Theorem 2, we can obtain such ideals in a ring of the Krull dimension 3 (resp. 4).

References

[Rat76] Ratliff, Jr., L. J., On prime divisors of I^n, n large. *Michigan Mathematical Journal* 23 : 337-352, 1976.

[Rat83] Ratliff, Jr., L. J., A brief survey and history of asymptotic prime divisors. *Rocky Mountain Journal of Mathematics* 13 : 437-459, 1983.

[Brod79] Brodmann, M., Asymptotic stability of $\mathrm{Ass}(M/I^nM)$. *Proceedings of the American Mathematical Society* 74 : 16-18, 1979.

[HNTT21] Hà, Huy Tài and Nguyen, Hop Dang and Trung, Ngo Viet and Trung, Tran Nam, Depth functions of powers of homogeneous ideals *Proceedings of the American Mathematical Society* 149 : 1837-1844, 2021.

[HeHi05] Herzog, Jürgen and Hibi, Takayuki, The depth of powers of an ideal *Journal of Algebra* 291 : 534-550, 2005.

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