F-representation type of invariant subrings of finite groups
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This is a joint work with Anurag Singh. Let k be a perfect field of prime characteristic p. For a k-vector space W and $e \in \mathbb{Z}$, the eth Frobenius twist ${}^eW$ of W is the additive group $\{{}^ew \mid w \in W\}$ whose addition is given by ${}^ew_1 + {}^ew_2 = {}^e(w_1 + w_2)$. We endow a k-action on ${}^eW$ by $\alpha \cdot {}^ew = {}^e(\alpha^{p^e} w)$. This makes ${}^eW$ a k-vector space. If B is a commutative k-algebra, then letting ${}^e(b_1 b_2) = {}^eb_1 {}^eb_2$, ${}^eB$ is a commutative k-algebra whose underlying ring is isomorphic to B. If, moreover, B is finitely generated over k, then so is ${}^eB$. If, moreover, $B = \bigoplus_{i \geq 0} B_i$ is a positively graded $\mathbb{Z}$-algebra, then letting $\deg {}^eb = i/p^e$ for $b \in B_i$. ${}^eB$ is a positively $p^{-e}\mathbb{Z}$-graded algebra. Let G be a finite group, and W be a G-module. Then letting G act on ${}^eW$ by $g{}^ew = {}^e(gw)$, ${}^eW$ is a G-module. If $w_1, \ldots, w_n$ is a k-basis of W, then ${}^ew_1, \ldots, {}^ew_n$ is a k-basis of ${}^eW$, and if (a_{ij}) is the matrix of g on W with respect to the basis $w_1, \ldots, w_n$, then $(a_{ij}^{p^{-e}})$ is the matrix of g on ${}^eW$ with respect to the basis ${}^ew_1, \ldots, {}^ew_n$.

Let B be a finitely generated positively graded commutative k-algebra. Let e be a non-negative integer. Then the eth Frobenius map $F^e : B \to {}^eB$ is a degree-preserving finite k-algebra map. Let M be a $\mathbb{Q}$-graded finite B-module. Then ${}^eM$ is a $\mathbb{Q}$-graded finite ${}^eB$-module by ${}^eb{}^em = {}^e(bm)$ and $\deg {}^em = i/p^e$ for $m \in M_i$. So ${}^eM$ is also a $\mathbb{Q}$-graded finite B-module. By the Krull–Schmidt theorem, ${}^eM$ is uniquely a finite direct sum of indecomposable $\mathbb{Q}$-graded finite B-modules. Let $\mathcal{N}$ be a finite set of indecomposable $\mathbb{Q}$-graded B-modules. We say that M has *finite F-representation type* (in short, FFRT) by $\mathcal{N}$ if for each $e \geq 0$, there is a decomposition ${}^eM \cong N_{e,1} \oplus \cdots \oplus N_{e,s_e}$ such that $N_{e,j}(c_{e,j})$ is isomorphic to an element of $\mathcal{N}$ for each j ($1 \leq j \leq s_e$) and some $c_{e,j} \in \mathbb{Q}$, where $N(c)$ denotes the shift of degree of N by c. The notion of FFRT was originated by K. Smith and Van den Bergh [SvdB].

Let V be a finite-dimensional k-vector space, and G be a finite subgroup of $\mathrm{GL}(V)$ without pseudo-reflection. Let $S = \operatorname{Sym} V$ be the symmetric algebra of V, and $A = S^G$. In [SvdB], Smith and Van den Bergh proved that if the order $|G|$ of G is not divisible by $\operatorname{char} k$ (non-modular case), then A has FFRT. In fact, they proved that if B has FFRT, then its direct summand graded subalgebra has FFRT (as S has FFRT by $\{S\}$ and A is a direct summand of S in non-modular case, this proves the FFRT property of A).

In the modular case (the case that $\operatorname{char} k = p > 0$ divides $|G|$), not so many things have been known about the FFRT property of $A = S^G$.

We say that V is a permutation (resp. monomial) representation of G if there exists some k-basis $v_1, \ldots, v_n$ of V and a group homomorphism $h : G \to \mathfrak{S}_n$ such that $gv_i = v_{h(g)(i)}$ (resp. $k \cdot (gv_i) = k \cdot v_{h(g)(i)}$) for $g \in G$ and $1 \leq i \leq n$. Note that a permutation representation is a monomial representation.

Theorem 1. *If V is a monomial representation of G, then the ring of invariants $A = (\operatorname{Sym} V)^G$ has FFRT.*

There has been no known examples of V such that A does not have FFRT. In this talk, we show that if $p = \operatorname{char} k = 3$, and k is transcendental over the prime field $\mathbb{F}_3$, then there is an example of G and V such that A has FFRT.

Let $G = C_3 \times C_3 = \langle \sigma, \tau \rangle$ be the elementary abelian group of order 9, where $\sigma^3 = \tau^3 = \sigma\tau\sigma^{-1}\tau^{-1} = e$. For $\alpha \in k$, let $V(\alpha) = k^3$ be the three-dimensional G-module given by

$$\sigma \mapsto E + N = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \tau \mapsto E + \alpha N = \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{bmatrix},$$

where

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then ${}^e(V(\alpha)) \cong V(\alpha^{p^{-e}})$ for $e \in \mathbb{Z}$.

Let $\pi : \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ be the projection. Let $\mathbb{Q}$-grmod(G, S) denote the category of $\mathbb{Q}$-graded S-finite (G, S)-modules, and $M \in \mathbb{Q}$-grmod(G, S). For $h \in \mathbb{Q}/\mathbb{Z}$, we define $M_{\langle h \rangle} := \bigoplus_{a \in \pi^{-1}(h)} M_a$. Note that $M = \bigoplus_{h \in \mathbb{Q}/\mathbb{Z}} M_{\langle h \rangle}$ is a direct decomposition of M in $\mathbb{Q}$-grmod(G, S). For $a \in \mathbb{Q}$, we simply denote $M_{\langle \pi(a) \rangle}$ by $M_{\langle a \rangle}$. If $M \neq 0$, we define $\mathrm{LD}(M) := \min\{i \in \mathbb{Q} \mid M_i \neq 0\}$, and $\mathrm{LRep}(M) := M_{\mathrm{LD}(M)}$. If $\mathrm{LRep}(M)$ is an indecomposable G-module, then there is an indecomposable summand N of M, which is unique up to isomorphisms, such that $\mathrm{LD}(N) = \mathrm{LD}(M)$ and $\mathrm{LRep}(N) \cong \mathrm{LRep}(M)$. We denote this N by $\mathrm{LInd}(M)$.

Theorem 2. *Let k and G be as above, t be transcendental over $\mathbb{F}_3$, and $V = V(t)$. Let $S = \operatorname{Sym} V$ be the symmetric algebra, and $A = S^G$. Then A does not have FFRT.*

Let $\mathbb{Q}$-grmod$A$ denote the category of $\mathbb{Q}$-graded finite A-modules. Then $M \mapsto M^G$ gives a functor $\mathbb{Q}$-grmod$(G, S) \to \mathbb{Q}$-grmod$A$, and it induces an equivalence ${}^*\operatorname{Ref}(G, S) \to {}^*\operatorname{Ref} A$ between the full subcategories consisting of reflexive modules. For $e \geq 0$, $({}^eS)_{1/p^e} = {}^e(S_1) = {}^eV = V(t^{1/p^e})$, and

$${}^0V = V(t), {}^1V = V(t^{1/p}), {}^2V = V(t^{1/p^2}), \ldots$$

are all different. So

$$\mathrm{LInd}(S_{\langle 1 \rangle}), \mathrm{LInd}(({}^1S)_{\langle 1/p \rangle}), \mathrm{LInd}({}^2S)_{\langle 1/p^2 \rangle}), \ldots$$

are all different, and they are indecomposable direct summands of ${}^eS$ for some e. So

$$\mathrm{LInd}(S_{\langle 1 \rangle})^G, \mathrm{LInd}(({}^1S)_{\langle 1/p \rangle})^G, \mathrm{LInd}({}^2S)_{\langle 1/p^2 \rangle})^G, \ldots$$

are all different, and they are indecomposable direct summands of ${}^eA$ for some e, and this proves the theorem.

References

[SvdB] K. E. Smith and M. Van den Bergh, Simplicity of rings of differential operators in prime characteristic, *Proc. London Math. Soc.* (3) **75** (1997), 32–62.