REFLEXIVE MODULES OVER AUSLANDER-GORENSTEIN RINGS

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A finitely generated module M over a commutative Noetherian ring R is called *reflexive* if the evaluation map $M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R)$ is an isomorphism. Reflexive modules (and their relatives) play a prominent role in representation theory [1, 3, 4], non-commutative resolutoions [11, 9, 10], and so on.

The main object of this talk is the category $\mathrm{ref}\, R$ of reflexive R-modules. We need some constraints on the ring R for $\mathrm{ref}\, R$ behave well. One well-established sufficient condition is that R should be normal. For example, it would imply that $\mathrm{ref}\, R$ is exactly the category of second syzygies, and that the inclusion $\mathrm{ref}\, R \to \mathrm{mod}\, R$ to the category of finitely generated modules has a left adjoint.

We shall study the category $\mathrm{ref}\, R$ not only over a commutative ring R, but also on an arbitrary two-sided Noetherian ring Λ, which allows us to cover some important representation theoretic examples including (higher) Auslander algebras as well as commutative Noetherian normal domains.

The aim of the talk is to point out that the nice behaviors of the category $\mathrm{ref}\, \Lambda$ is governed by the *Auslander-type conditions*, namely, the conditions on the minimal injective resolution of the ring Λ. Let

$$0 \longrightarrow \Lambda \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

be the one. For $l, n \in \mathbb{Z}_{>0}$, we say that Λ satisfies the (l, n)*-condition* [6] if $\mathrm{flat.dim}\, I^i < l$ for each $0 \leq i < n$. These appear in various representation theoretic problems e.g. [2, 5, 3, 4, 7, 8], and it is through this type of condition that the seemingly different class of algebras as above can be treated in a unified setting.

Our main result is a characterization of the category $\mathrm{ref}\, \Lambda$ being quasi-abelian in terms of Auslander-type conditions.

Theorem. *Let Λ be a two-sided Noetherian ring and $\mathcal{C} = \mathrm{ref}\, \Lambda$ be the category of reflexive Λ-modules. Then the following are equivalent.*

(i) *The category $\mathcal{C}$ is quasi-abelian.*
(ii) *The rings Λ and Λ^{op} satisfy the $(2, 2)$-condition.*

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