Face numbers of (high dimensional) triangulated mfds

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Combinatorial Structures in Geometric Topology

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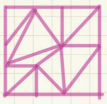
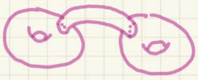




Aim of the talk

Explain some ideas to study face numbers of high dim triangulated mfds that comes from algebraic tools

おわび (apology)

I will not talk about "Stanley-Reisner theory"

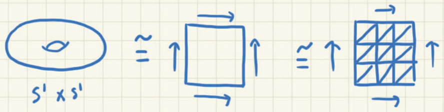
1. Quick intro. to face numbers of triangulated mfd.

Set up

- Δ : (finite abstract) simplicial complex
- $f_i(\Delta)$ = num. of i-dim faces of Δ

  $f(\Delta) = (f_0(\Delta), f_1(\Delta), \dots, f_d(\Delta))$ ← $d = \dim \Delta$
- Δ is a triangulation of X

  $\Leftrightarrow |\Delta| \cong_{homeo} X$ — $|\Delta|$: geometric realization



← this triangulation: 9 vertices, 27 edges, 18 triangles, so $f(\Delta) = (9, 27, 18)$

- M : connected closed mfd admitting a triangulation

Main theme of this talk

Problem Fix a mfd M. Find nice bounds for face numbers of triangulations of M.

Related Problems

1. How many vertices do we need to triangulate a mfd.

   More precisely determine $f_0^{\min}(M) := \min\{f_0(\Delta) \mid \Delta \text{ triangulates } M\}$
2. Can we determine

   $f_i^{\max}(\Delta, n) = \max\{f_i(\Delta) \mid \Delta \text{ is an } n\text{-vertex triang. of } M\}$
3. Can we determine

   $f_i^{\min}(\Delta, n) = \min\{f_i(\Delta) \mid \Delta \text{ is an } n\text{-vertex triang. of } M\}$

Low dim case : 2-mfds (surfaces)

- M : connected closed surface
- Δ : triangulation of M

  $\Rightarrow f(\Delta) = (f_0, 3(f_0 - \chi(M)), 2(f_0 - \chi(M)))$

Why? $f = (f_0, f_1, f_2)$ but
- $f_0 - f_1 + f_2 = \chi(M)$
- $3f_1 = 2f_2$

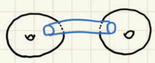
Theorem (Ringel 1955, Jungerman-Ringel 1980))
If M is a closed surface with $M \neq S_2, N_2, N_3$, then

$$f_0^{\min}(M) = \min\left\{ f_0 \;\middle|\; \binom{f_0 - 3}{2} \geq 3(2 - \chi(M)) \right\}$$

← Heawood inequality

- $S_g \overset{def}{=} (S^1 \times S^1)^{\# g} = \underbrace{(S^1 \times S^1) \# (S^1 \times S^1) \# \cdots \# (S^1 \times S^1)}_{g \text{ copies}}$
- $N_g \overset{def}{=} (\mathbb{RP}^2)^{\# g}$

\# means connected sum



$(S^1 \times S^1) \# (S^1 \times S^1)$

Low dim case : 3-mfds

- M : connected closed 3-mfds
- Δ : triangulation of M

  $\Rightarrow f(\Delta) = (f_0, f_1, 2(f_1 - f_0), f_1 - f_0)$

Why?
- $f_0 - f_1 + f_2 - f_3 = 0$
- $2f_2 = 4f_3$

$f(\Delta)$ only depends $f_0(\Delta)$ and $f_1(\Delta)$
(this is also true for 4-mfds)

Q Can we determine $f_0^{\min}(M)$?
$\Rightarrow$ sometimes possible but usually very hard

ex $f_0^{\min}(S^1 \times S^2) = 10$, $f_0^{\min}(S^2 \times S^2) = 11$, $f_0^{\min}(\mathbb{RP}^3) = 11$,
$f_0^{\min}(S^1 \times S^1 \times S^1) = 15$? (probably open)

Low dim case : 3-mfds

- M : connected closed mfd of dim $d \geq 3$

Thm (Walkup 1970, Swartz 2009)
$f_1^{\max}(M, n) = \binom{n}{2}$ for $n \gg 0$

Thm (Novik-Swartz 2009)
$f_1^{\min}(M, n) \geq (d+1)n + \binom{d+2}{2}(\beta_1(M) - 1)$

※ $\beta_i(M)$ = ith Betti numbers computed over $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$

Complete description

A complete description of $f(\Delta)$ is obtained for following mfds

① $S^3, S^2\times S^1, S^2 \tilde{\times} S^1, \mathbb{R}P^3$ (Walkup 1970)

② $(S^2\times S^1)^{\#2}, S^3\times S^1, \mathbb{C}P^2$ (Swartz 2009)

③ $S^3 \tilde{\times} S^1$ (Chestnut-Sapir-Swartz 2008)

④ $(S^2\times S^1)^{\#k}$ and $(S^2\tilde{\times} S^1)^{\#k}$ for small k (Lutz-Sulanke-Swartz 2009)

⑤ S^d for any $d \ge 4$ (Billera-Lee 1981 (sufficiency); Stanley 1980, Adiprasito 2018, Papadakis-Petrotou 2020 (necessity))

How we study face numbers of mfds?

- For surfaces, we only need to consider "f_0"
- For 3- and 4-mfds, we need to consider "f_0, f_1". In particular, it is important to understand $f_0^{min}(M)$, $f_1^{min}(M,n)$, $f_1^{max}(M,n)$
- For mfds of dim $d \ge 5$, we need to understand "$f_0, f_1, \dots, f_{\lfloor d/2 \rfloor}$"

Q How we can study this?

2. Face numbers of higher dim mfds

Goal explain an idea to study $f(\Delta)$ of high dim triangulated mfd Δ that comes from algebraic study (but I will not go into algebra)

the case of a sphere

Point We want to understand "$f(\Delta)$". But "$f(\Delta)$" is not a right object to study.

Def Δ: triangulation of S^{d-1}. Define $h_0(\Delta), h_1(\Delta), \dots, h_d(\Delta)$ by

$$h_k(\Delta) := \sum_{i=0}^{k} (-1)^{k-i}\binom{d-i}{d-k} f_{i-1}(\Delta)$$

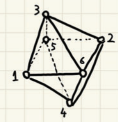
← we set $f_{-1}(\Delta)=1$

These "h"-numbers have the following properties

① Knowing $f_0(\Delta), \dots, f_{d-1}(\Delta)$ is equivalent to $h_0(\Delta), \dots, h_d(\Delta)$

② $h_i(\Delta) = h_{d-i}(\Delta)$ (Dehn-Sommerville equation)

③ $h_0 \le h_1 \le \cdots \le h_{\lfloor d/2 \rfloor} \ge \cdots \ge h_d$ (Stanley, Adiprasito, Papadakis-Petrotou)



$f=(6,12,8)$

⇓

$h=(h_0,h_1,h_2,h_3)=(1,3,3,1)$

the case of a sphere

Δ: triangulation of S^{d-1}

Define $g_0(\Delta), g_1(\Delta), \dots, g_{\lfloor d/2 \rfloor}(\Delta)$ by

$$g_i(\Delta) = h_i(\Delta) - h_{i-1}(\Delta)$$

Rem Knowing h numbers is equivalent to knowing g numbers

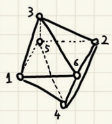
g-theorem (Billera-Lee, Stanley, Adiprasito, Papadakis-Petrotou)

$g=(g_0=1, g_1, \dots, g_{\lfloor d/2 \rfloor})$: seq. of non-negative integers. TFAE

(1) $\exists$ triangulation Δ of S^{d-1} s.t. $g_k = g_k(\Delta)$ for all k

(2) $\exists$ homogeneous ideal $I \subset S = \mathbb{R}[x_1, \dots, x_{g_1}]$ s.t. $g_k = \dim_{\mathbb{R}}(S/I)_k$ for all k

↑ homogeneous component of degree k



$f=(6,12,8)$

⇓

$h=(h_0,h_1,h_2,h_3)=(1,3,3,1)$

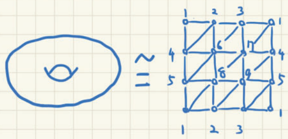
⇓

$g=(1,2)$

the case of mfds

Point "$h(\Delta)$ and $g(\Delta)$" are not right object to study

example



This has $f(\Delta)=(9,27,18)$

⇓

$h=(1,6,12,-1)$

Q Maybe there is a better object to study?

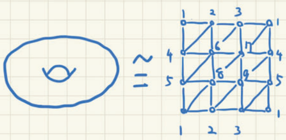
the case of mfds

Δ: triangulation of a connected closed $(d-1)$-mfd M

Def Define $h''_0(\Delta), h''_1(\Delta), \dots, h''_d(\Delta)$ by

$$h''_i(\Delta) = h_i(\Delta) - \binom{d}{i}\Bigl(\sum_{\substack{1\le k\le i \\ k\neq d}} (-1)^{i-k}\beta_{k-1}(M)\Bigr)$$

example



$f(\Delta)=(9,27,18)$

⇓

$h=(1,6,12,-1)$

⇓

$h''=h-2\times(0,0,3,-1)$
$=(1,6,6,1)$ ← Symmetric!

the case of mfds

Δ: triangulation of a connected closed $(d-1)$-mfd M

Def Define $h''_0(\Delta), h''_1(\Delta), \dots, h''_d(\Delta)$ by

$$h''_i(\Delta) = h_i(\Delta) - \binom{d}{i}\Bigl(\sum_{\substack{1\le k\le i \\ k\neq d}} (-1)^{i-k}\beta_{k-1}(M)\Bigr)$$

These h''-numbers have the following properties

① Knowing $f_0(\Delta), \dots, f_{d-1}(\Delta)$ is equivalent to $h''_0(\Delta), \dots, h''_d(\Delta)$ ← assuming that we know $\beta_k(M)$

② $h''_i(\Delta) = h''_{d-i}(\Delta)$ (Novik 1998 (essentially Klee 1964))

③ $h''_0 \le h''_1 \le \cdots \le h''_{\lfloor d/2 \rfloor} \ge \cdots \ge h''_d$ (Adiprasito-Papadakis-Petrotou 2021)

the case of mfds

Δ: triangulation of a connected closed (d-1)-mfd M

Define $\tilde g_0(\Delta), \tilde g_1(\Delta), \dots, \tilde g_{\lfloor \frac{d}{2} \rfloor}(\Delta)$ by

$$\tilde g_i(\Delta) = h_i(\Delta) - h_{i-1}(\Delta) - \binom{d+1}{i}\left(\sum_{k=2}^{i}(-1)^{i-k}\beta_{k-1}(M)\right)$$

Thm (M-Nevo 2014 (+APP))

There is a homogeneous ideal $J \subset S = \mathbb{R}[x_1,\dots,x_{\tilde g_1}]$ s.t.

(1) $\tilde g_k(\Delta) = \dim_{\mathbb{R}}(S/J)_k$ for all k

(2) $\dim_{\mathbb{R}} J_k \ge \binom{d+1}{k}\beta_{k-1}(M)$

Remark

$\tilde g_i \ne h''_i - h''_{i-1}$

APP means

Adiprasito 2018, Papadakis-Petrotou 2020

Quick summery

☑ For triangulations Δ of S^{d-1}, we think that h-numbers g-numbers behave much nicer than f(Δ)

$g_k = \dim_{\mathbb{R}}(S/I)_k$ for all k $\exists I$: ideal of $S=\mathbb{R}[x_1,\dots,x_g]$

☑ For triangulations Δ of a (d-1)-mfd, from an algebraic view point h''-numbers and $\tilde g$-numbers look to be right object to study

• $\tilde g_k = \dim_{\mathbb{R}}(S/J)_k$ for all k

• $\dim_{\mathbb{R}} J_k \ge \binom{d+1}{k}\beta_{k-1}(M)$

$\exists J$: ideal of $S=\mathbb{R}[x_1,\dots,x_{\tilde g_1}]$

Applications of algebraic ideas

A generalization of Heawood inequality (Novik-Swartz 2009 +APP)

If Δ is a triangulation of a connected closed (d-1)-mfd, then

$$\binom{f_0(\Delta)-d-2-r}{r} \ge \binom{d+1}{r}\beta_{r-1}(\Delta) \quad \left(r<\frac{d-1}{2}\right)$$

originally conjectured by Kühnel

Rem there is a $r=\frac{d-1}{2}$ version of the thm

The r=2 case of the thm is applied to show

$f_0^{\min}((S^3\times S^1)^{\#143})=71$ $f_0^{\min}((S^3\times S^1)^{\#342})=101$

$f_0^{\min}((S^2\underline{\times} S^1)^{\#208})=69$ $f_0^{\min}((S^2\underline{\times} S^1)^{\#546})=109$

(Burton-Datta-Singh-Spreer 2018)

Applications of algebraic ideas

A generalization of Heawood inequality (Novik-Swartz 2009 +APP)

If Δ is a triangulation of a connected closed (d-1)-mfd, then

$$\binom{f_0(\Delta)-d-2-r}{r} \ge \binom{d+1}{r}\beta_{r-1}(\Delta) \quad \left(r<\frac{d-1}{2}\right)$$

originally conjectured by Kühnel

Pf) We know $\exists J \subset S = \mathbb{R}[x_1,\dots,x_{\tilde g_1}]$ s.t.

☑ ~~$\tilde g_k(\Delta) = \dim_{\mathbb{R}}(S/J)_k$~~

☑ $\binom{d+1}{r}\beta_{r-1}(\Delta) \le \dim_{\mathbb{R}} J_r \le \dim_{\mathbb{R}} S_r = \binom{\tilde g_1+r-1}{r} = \binom{f_0-d-2+r}{r}$ □

Note

$\tilde g_1 = f_0-d-1$

Applications of algebraic ideas

M: connected closed d-mfd

Thm (Novik-Swartz 2009)

$$f_1^{\min}(M,n) \ge (d+1)n + \binom{d+2}{2}(\beta_1(M)-1)$$

This is essentially $\tilde g_2(\Delta)\ge 0$

This is a statement that I introduced in part 1

The theorem is applied to get a complete characterization of face numbers of $(S^2\times S^1)^{\#k}$ for small k

When equality holds?

M: connected closed d-mfd

Δ: a triangulation of M

Thm (M-Nevo 2014 +APP)

$\tilde g_r(\Delta)=0 \Rightarrow \beta_k(\Delta)=0$ for $k\ne 0, r-1, d-r+1, d$

Rem Equality in Generalized Heawood inequality for r ⇒ $\tilde g_r(\Delta)=0$

Meaning of Thm A generalization of Heawood inequality (and $\tilde g_i\ge 0$) might be sharp only for very special mfd like $(S^i\times S^j)^{\#k}$

Summary and Problems

☑ I explained that $\tilde g$-numbers look to give a right object to study face numbers of mfds

☑ $\tilde g$-numbers works well for $(S^{d-1}\times S^1)^{\#k}$ (probably also for $(S^i\times S^j)^{\#k}$) but we probably need new ideas for mfds like $\mathbb{RP}^d$, $T^d=S^1\times\cdots\times S^1$ etc.

For algebraists

nice combinatorial conjecture → Find its algebraic meaning → Feedback to combinatorial problems

we need this

Some problems (that I am interested in)

Face numbers of $S^{d-1}\times S^1$ (and $S^{d-1}\underline{\times} S^1$)

• We know all possible face numbers of S^d ← $g_2=0$

• We also know $f_i^{\min}(S^{d-1}\times S^1,n)$ ← given by stacked triangulations

Q Can we determine $f_i^{\max}(S^{d-1}\times S^1,n)$?

My conj.

(1) $\tilde g_i^{\max}(S^{d-1}\times S^1,n) \le \dim_{\mathbb{R}}\left(\mathbb{R}[x_1,\dots,x_{\tilde g_1}]/(x_1,\dots,x_{d+1})^2\right)_i$

(2) $\exists\Delta\cong S^{d-1}\times S^1$ s.t. $\tilde g_i(\Delta)=\dim_{\mathbb{R}}\left(\mathbb{R}[x_1,\dots,x_{\tilde g_1}]/(x_1,\dots,x_{d+1})^2\right)_i$ $(\forall i)$

Motivation

$\exists J\subset S=\mathbb{R}[x_1,\dots,x_{\tilde g_1}]$ s.t.

☑ $\tilde g_i(\Delta)=\dim_{\mathbb{R}}(S/J)_i$

☑ $\dim_{\mathbb{R}} J_2 \ge \binom{d+2}{2}\beta_1$

max is given by $J=(x_1,\dots,x_{d+1})^2$?

Some problems (that I am interested in)

Face numbers of d-torus $T^d = S^1 \times \cdots \times S^1$

On triangulation of T^d, we have an interesting conj.

Conj. (Lutz) $f_0^{min}(T^d) = 2^d - 1$

Rem "$\leq$" is known

Q Can we give a good guess on $f_\lambda^{min}(T^d, n)$ or $\tilde{g}_\lambda^{min}(T^d, n)$?

Some problems (that I am interested in)

Simplicial cell complex

- A simplicial cell complex is a regular CW cpx s.t. the boundary of each cell is combinatorially isom. to the boundary of a simplex
- h''-numbers look to be a right object to study
- complete characterization of $f(\Delta)$: S^d, $\mathbb{R}P^d$, $S^i \times S^j$, B^d (Stanley 91, Masuda 05, M 13)

Q $f_d \geq (d+1)!$ for d-torus $T^d = S^1 \times S^1 \times \cdots \times S^1$

Q $f_1 - 4f_0 + 6 \geq 22$ for 3-torus?

$\uparrow$ $h''_2 \geq 4$

Thank you for your attention!!